Problem Set 10 (Due Tuesday, March 19)

Non-book problems:
1. Let \( \alpha = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\} \) and \( \beta = \{(1, 1), (-1, 1)\} \) be (ordered) bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \), respectively. Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be a linear transformation whose matrix with respect to \( \alpha \) and \( \beta \) is

\[
[T]_\beta^\alpha = \begin{bmatrix} 1 & 0 & 4 \\ -1 & -3 & 2 \end{bmatrix}.
\]

1a. Work out a formula for \( T((x_1, x_2, x_3)) \).

1b. Find the matrix of \( T \) with respect to the standard bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \).

2. Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be the linear transformation given by

\[
T((x, y)) = (2x - y, x + 3y, -y).
\]

2a. Write down the matrix of \( T \) with respect to the standard bases \( \alpha = \{\vec{e}_1, \vec{e}_2\} \) of \( \mathbb{R}^2 \), and \( \beta = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) of \( \mathbb{R}^3 \).

2b. Let \( \alpha' = \{\vec{e}_2, \vec{e}_1\} \) be the standard basis of \( \mathbb{R}^2 \), but in the opposite order. Calculate the matrix \( [T]_\beta^{\alpha'} \).

2c. Let \( \beta' = \{\vec{e}_1, \vec{e}_3, \vec{e}_2\} \) be the standard basis of \( \mathbb{R}^3 \), but with the last two vectors swapped. Calculate the matrix \( [T]_\alpha^{\beta'} \).

2d. Now let \( V,W \) be any finite-dimensional vector spaces, and let \( T : V \to W \) be a linear transformation. Let \( \alpha \) be an ordered basis of \( V \), and let \( \beta \) be an ordered basis of \( W \). Based on your answers to parts (a)-(c), write down a general rule for what happens to the matrix \( [T]_\alpha^\beta \) when you (i) swap two of the vectors in the basis \( \alpha \), or (ii) swap two of the vectors in the basis \( \beta \).

(You do not need to prove that your rule is correct.)

Book problems:
2.3, #1(a,f), 3(b,c), 7(a), 13

[For #3, assume the “vector spaces of appropriate dimension” are just \( \mathbb{R}^n \) for the appropriate \( n \). Also, note that #13 has three parts.]

Optional Challenges (Do not hand in): 2.3, #4, 7(b,c), 8, 10(a), 14

See the reverse side of this sheet for some suggestions and hints.
Suggestions and Hints on Kernel and Image Bases

Several of the problems in Section 2.3 ask you to find bases for the kernel and the image of a linear map \( T : V \to W \). Methods for doing this are discussed in the book on pages 86–91, in a sequence of propositions, examples, and discussions. Here is a quick summary of the method:

1. Find bases \( \alpha = \{ \vec{v}_1, \ldots, \vec{v}_n \} \) for \( V \) and \( \beta = \{ \vec{w}_1, \ldots, \vec{w}_m \} \) for \( W \), and compute the matrix \( A = [T]_\beta^\alpha \).
   [Note: in some cases, some or all of this step may already be done for you. The point is to get a matrix \( A \) to work with, to represent \( T \).]

2. Writing
   \[
   A = \begin{bmatrix}
   a_{11} & \cdots & a_{1n} \\
   \vdots & \ddots & \vdots \\
   a_{m1} & \cdots & a_{mn}
   \end{bmatrix},
   \]
   do row reduction to solve the system
   \[
   \begin{bmatrix}
   a_{11} & \cdots & a_{1n} & 0 \\
   \vdots & \ddots & \vdots & \vdots \\
   a_{m1} & \cdots & a_{mn} & 0
   \end{bmatrix}
   \]

3. You will get that \( \text{Ker}(A) = \text{Span}\{ \vec{x}_1, \ldots, \vec{x}_k \} \subseteq \mathbb{R}^n \) where \( k \) is the number of free variables, and each \( \vec{x}_i = (c_{i,1}, \ldots, c_{i,n}) \) belongs to \( \mathbb{R}^n \).

4. If \( V = \mathbb{R}^n \) and \( \alpha \) is the standard basis \( \{ \vec{e}_1, \ldots, \vec{e}_n \} \), then \( \{ \vec{x}_1, \ldots, \vec{x}_k \} \) is the desired basis for \( \text{Ker}(T) = \text{Ker}(A) \). Otherwise, to get a basis for \( \text{Ker}(T) \), don’t forget to use the basis \( \alpha \) to produce \( \{ \vec{u}_1, \ldots, \vec{u}_k \} \subseteq V \), where \( \vec{u}_i = c_{i,1} \vec{v}_1 + \cdots + c_{i,n} \vec{v}_n \).

5. Go back and look at the echelon form you got when you did the above row reduction, and see which columns the pivots are in. Then list the corresponding columns \( \vec{y}_1, \ldots, \vec{y}_r \in \mathbb{R}^m \) chosen from the original matrix \( A \).

6. If \( W = \mathbb{R}^m \) and \( \beta \) is the standard basis \( \{ \vec{e}_1, \ldots, \vec{e}_m \} \), then \( \{ \vec{y}_1, \ldots, \vec{y}_r \} \) is the desired basis for \( \text{Im}(T) = \text{Im}(A) \). Otherwise, to get a basis for \( \text{Im}(T) \), don’t forget to use the basis \( \beta \) to produce \( \{ \vec{z}_1, \ldots, \vec{z}_k \} \subseteq V \), where \( \vec{z}_i = d_{i,1} \vec{w}_1 + \cdots + d_{i,m} \vec{w}_m \), with \( \vec{y}_i = (d_{i,1}, \ldots, d_{i,m}) \).

The key facts in points 5 and 6 above are:

- The image \( \text{Im}(A) \), which is a subspace of \( \mathbb{R}^m \), is equal to the span of the columns of \( A \). (This is why the image \( \text{Im}(A) \) of a matrix is sometimes called the columnspace of \( A \).

- Every column of \( A \) can be written as a linear combination of just those columns of \( A \) which ultimately will have a pivot in them after you do row reduction.

Warning: do NOT pull the pivot columns from the echelon form itself; instead, use the echelon form to figure out which columns of the original matrix \( A \) to use.

And of course, as I said in two places above, if \( V \) or \( W \) is not \( \mathbb{R}^n \) or \( \mathbb{R}^m \) with the standard basis, don’t forget to convert your answers back to elements of the actual vector spaces.