Midterm Exam 1, Friday, March 1, 2019

Instructions: Do all seven numbered problems. If you wish, you may also attempt the optional bonus problem. Show all work, including scratch work. Little or no credit may be awarded, even when your answer is correct, if you fail to follow instructions for a problem or fail to justify your answer. If you need more space, use the back of any page. If you have time, check your answers.

WRITE LEGIBLY. NO CALCULATORS.

1. (17 points) Find the set of all solutions \((x, y, z, w) \in \mathbb{R}^4\) to the following system of equations.
   \[
   \begin{align*}
   x - 2y + 3z - 2w & = 2 \\
   2x + y + z + 6w & = 14 \\
   x + y + 4w & = 8
   \end{align*}
   \]

2. (12 points) Let \(V\) be a vector space, and let \(S = \{\vec{x}_1, \ldots, \vec{x}_n\} \subseteq V\) be a finite subset. Define the following terms and phrases. You may use other standard terms without defining them.
   2a. \(\text{Span}(S)\).
   2b. \(S\) is linearly independent.

3. (15 points) Prove the following theorem we have learned: Let \(V\) be a vector space, and let \(W_1, W_2 \subseteq V\) be subspaces of \(V\). Then \(W_1 \cap W_2\) is a subspace of \(V\).

4. (12 points) Working in the vector space \(\mathbb{R}^4\), let \(S = \{(2, 1, 0, 3), (0, 1, -1, 2)\}\).
   Is \((4, 3, 2, 1) \in \text{Span}(S)\)? Why or why not?

5. (15 points) Let \(V = F(\mathbb{R})\), the vector space of functions from \(\mathbb{R}\) to \(\mathbb{R}\).
   Let \(W = \{f \in V \mid f(5) = 4f(2)\}\). Prove that \(W\) is a subspace of \(V\).

6. (17 points) Let \(V = P_2(\mathbb{R})\), the vector space of polynomials of degree at most 2.
   Let \(W = \{p \in V : p(2) = 2p'(0)\}\).
   It is a fact, which you may assume, that \(W\) is a subspace of \(P_2(\mathbb{R})\).
   Find a basis for \(W\).

7. (12 points) In this problem, \(a, b, c \in \mathbb{R}\) are three specific numbers that I am keeping secret.
   7a. Consider the set \(S_1 = \left\{ \begin{bmatrix} 0 \\ 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 7 \\ c \end{bmatrix} \right\}\) of three vectors in \(\mathbb{R}^4\).
   Take my word for it that \(S_1\) is linearly independent. Is \(\text{Span}(S_1) = \mathbb{R}^4\)?
   Answer “Yes,” “No,” or “Need more information.” Justify your answer.

   7b. Consider the set \(S_2 = \left\{ \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 6 \\ c \end{bmatrix} \right\}\) of four vectors in \(\mathbb{R}^4\).
   Take my word for it that \(\text{Span}(S_2) = \mathbb{R}^4\). Is \(S_2\) linearly independent?
   Answer “Yes,” “No,” or “Need more information.” Justify your answer.
OPTIONAL BONUS. (2 points) Let $V$ be a vector space, let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in V$, and let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Suppose $\dim(V) = 12$, and let

$$S = \{a_1 \vec{v}_1 + a_2 \vec{v}_2, \ a_2 \vec{v}_1 + a_3 \vec{v}_3, \ a_4 \vec{v}_2 + a_1 \vec{v}_4, \ a_1 \vec{v}_1 + a_4 \vec{v}_4, \ a_4 \vec{v}_3 + a_2 \vec{v}_4\}.$$ 

Suppose that the (five) elements of $S$ are all distinct. Prove that the set $S$ is linearly dependent.