Some Useful Eigenvalue Facts

In general, if you want to find the eigenvalues of an \( n \times n \) matrix \( A \), you will need to compute the characteristic polynomial and find its roots. Sometimes, however, you can find an eigenvalue or two with minimal effort. This handout describes some tricks which sometimes work, as well as some general facts that can be useful.

**Fact 1:** Let \( A \) be an upper-triangular or lower-triangular matrix. Then the eigenvalues of \( A \) are precisely the diagonal entries of \( A \).

(This is Exercise 4.1.11.)

**Example.** The eigenvalues of \( A = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix} \) are 3, 3, -1, i.e., \( \lambda_1 = 3 \) with algebraic multiplicity 2, and \( \lambda_2 = -1 \) with algebraic multiplicity 1. [Incidentally, this matrix is not diagonalizable, because a simple computation shows that the eigenspace \( E_3 = \text{Ker}(A-3I) \) has dimension 1. Thus, \( \lambda_1 = 3 \) has algebraic multiplicity 2 but geometric multiplicity only 1, so \( A \) is not diagonalizable.]

**Fact 2:** If \( A \in M_{n \times n}(\mathbb{R}) \) has a nontrivial kernel (equivalently, \( A \) is not invertible; equivalently, \( \det(A) = 0 \)), then \( \lambda = 0 \) is an eigenvalue, and its eigenspace is \( \text{Ker}(A) \).

(This is immediate from the definitions)

**Fact 3:** If \( A \in M_{n \times n}(\mathbb{R}) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), then
\[
\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{Tr}(A), \quad \text{and} \quad \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A).
\]

(This is Exercise 4.1.7)

Note: The key insight behind Fact 3 is that on the one hand, we know from class that the characteristic polynomial \( p(t) = \det(A - tI) \) looks like
\[
p(t) = (-1)^n t^n + (-1)^{n-1} \text{Tr}(A)t^{n-1} + \cdots + \det(A)
\]
where \( \text{Tr}(A) \) denotes the trace of \( A \), i.e., the sum \( a_{11} + a_{22} + \cdots + a_{nn} \) of the diagonal entries of \( A \), and on the other hand, since its roots are precisely \( \lambda_1, \ldots, \lambda_n \), we have
\[
p(t) = (-1)^n(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n),
\]
which has \( t^{n-1} \)-coefficient \( (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n) \) and constant term \( \lambda_1 \cdots \lambda_n \).

**Example.** Let \( B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix} \). Since each row is a scalar multiple of the first, \( B \) has nontrivial kernel. In fact, row reduction gives the echelon form \( \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), and the fact that there are two free columns means that \( \text{dim}(\text{Ker}(B)) = 2 \). So \( \lambda = 0 \) is an eigenvalue of \( B \) with geometric multiplicity 2, and hence algebraic multiplicity at least 2.

But what is the third eigenvalue \( \lambda \)? Well, the trace of \( B \) is \( 1 + (-2) + (-2) = -3 \), so the sum of the eigenvalues \( 0 + 0 + \lambda \) must also be -3, i.e., \( \lambda = -3 \). So the eigenvalues of \( B \) are 0 (with algebraic and geometric multiplicity both 2), and 3 (with algebraic and geometric multiplicity both 1).

[And incidentally, \( B \) is diagonalizable, by Theorem 4.2.7.]
Fact 4: A and $A^t$ have the same characteristic polynomial, and hence the same eigenvalues.

[This is just because the transpose $(A - tI)^t$ of $A - tI$ is the same as $A^t - tI$, and hence \( \det(A^t - tI) = \det((A - tI)^t) = \det(A - tI) \).]

Fact 5: Suppose that all of the rows of $A \in M_{n \times n}(\mathbb{R})$ sum to the same number $s$, or that all of the columns of $A \in M_{n \times n}(\mathbb{R})$ sum to the same number $s$. Then $s$ is an eigenvalue of $A$.

Proof of Fact 5. If every row of $A$ sums to $s$, then multiplying by $[1 \ 1 \ \cdots \ 1]^t$ gives

$$
A = \begin{bmatrix}
1 & a_{11} & a_{12} & \cdots & a_{1n} \\
1 & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
= \begin{bmatrix}
1 & a_{11} + a_{12} + \cdots + a_{1n} \\
1 & a_{21} + a_{22} + \cdots + a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_{n1} + a_{n2} + \cdots + a_{nn}
\end{bmatrix}
= \begin{bmatrix}
s \\
s \\
\vdots \\
s
\end{bmatrix}
= \begin{bmatrix}
1 \\
s \\
\vdots \\
s
\end{bmatrix},
$$

showing explicitly that $s$ is an eigenvalue with eigenvector $[1 \ 1 \ \cdots \ 1]^t$.

If the columns of $A$ sum to $s$, then the rows of $A^t$ sum to $s$, so $s$ is an eigenvalue of $A^t$. By Fact 4, then, $s$ is also an eigenvalue of $A$.

QED Fact 5

Warning: If the rows of $A$ sum to $s$, the above proof gives an explicit associated eigenvector. But if the columns of $A$ sum to $s$, there’s no obvious eigenvector; you just have to do row reduction to compute $\text{Ker}(A - sI)$ as usual.

Example. Let $C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 1 & -3 & 5 & 1 \end{bmatrix}$. Since the second row is the same as the first, $C$ is not invertible, and so by Fact 2, 0 is an eigenvalue. In fact, a quick row reduction shows $C$ has nullity 2, so $\lambda = 0$ has eigenspace $E_0 = \text{Ker}(C)$ of dimension 2, and hence 0 has algebraic multiplicity at least 2.

In addition, every column of $C$ sums to 5. So 5 is also an eigenvalue, by Fact 5. Finally, summing the diagonal, we have $\text{Tr}(C) = 1 + 2 + 0 + 1 = 3$, so the eigenvalues sum to 3, by Fact 3. Already we know of 0, 0, 5 as eigenvalues, so the last eigenvalue must be $-2$.

So the eigenvalues of $C$ are $\lambda_1 = 0$ (with algebraic and geometric multiplicity 2), $\lambda_2 = 5$ (with algebraic and geometric multiplicity 1), and $\lambda_3 = -2$ (with algebraic and geometric multiplicity 1); and $C$ is diagonalizable.

Example. Fix $n \geq 2$, and let $M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ be the $n \times n$ matrix of all 1’s. A quick row reduction yields the echelon form

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix},
$$

which has 1 pivot and $n - 1$ free columns. So $\text{Ker}(M)$ is nontrivial, and so 0 is an eigenvalue of geometric multiplicity $\dim(\text{Ker}(M)) = n - 1$. The algebraic multiplicity of $0$ must therefore be at least $n - 1$.

Here are two different ways to find the $n$-th eigenvalue. First, we can see that $\text{Tr}(M) = n$ (summing the diagonal), so the last eigenvalue must be $n$, so that the eigenvalues $0, \ldots, 0, n$ sum to $n$. Second, we can use Fact 5 and the fact that every row of $M$ sums to $n$ to see that $n$ is an eigenvalue; this yields the bonus fact that the vector of all 1’s is an eigenvector of $M$ for the eigenvalue $n$.

So the eigenvalues of $M$ are $\lambda_1 = 0$ (with algebraic and geometric multiplicity $n - 1$), and $\lambda_2 = n$ (with algebraic and geometric multiplicity 1); and $M$ is diagonalizable.