Solutions to Homework #8

1. Section 2.3, #2
For this problem, you may give informal justifications (as opposed to formal proofs), but please mention where and how you are using the pigeonhole principle or related logic.
How many cards must be dealt from a standard deck of cards to guarantee:

(a) a pair, i.e., (at least) two cards of the same rank?

(b) a pair of aces?

(c) (at least) two cards of the same suit?

(d) five cards of the same suit?

Solutions. (a) $\boxed{\text{14 cards}}$ There are $m = 13$ ranks, so for the pigeonhole principle to guarantee n cards have two landing in the same rank, we need (at least) $n = m + 1 = 14$.

(b) $\boxed{\text{50 cards}}$ There are $52 - 4 = 48$ non-ace cards. Think of each of these 48 as being a single pigeonhole, and then one more pigeonhole for all four aces, giving a total of $m = 49$ pigeonholes. So for the pigeonhole principle to guarantee n cards have two landing in the same pigeonhole — and the only pigeonhole that can get more than one card is the one for the aces — we need (at least) $n = m + 1 = 50$.

Note: It's OK to explain this without explicit mention of the pigeonhole principle, something like the following. If one used only 49 or fewer cards, they could be all 48 non-aces, plus one of the aces, so 49 cards is not enough. But with 50 cards, there are only two left over, and those two leftovers can consist of at most two aces. So at least $4 - 2 = 2$ aces must be among the 50 cards that were dealt.

(c) $\boxed{\text{5 cards}}$ There are $m = 4$ suits, so for the pigeonhole principle to guarantee n cards have two landing in the same suit, we need (at least) $n = m + 1 = 5$.

(d) $\boxed{\text{17 cards}}$ There are $m = 4$ suits, so for the strong pigeonhole principle to guarantee n cards have at least $k = 5$ landing in the same suit, we need (at least) $n = 1 + m(k - 1) = 17$.

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2. Section 2.3, #3(a,b,c)
For this problem, you may give informal justifications (as opposed to formal proofs), but please mention where you and how are using the pigeonhole principle or related logic
A piggy bank contains 12 pennies, 8 nickels, 10 dimes, and 3 quarters. How many coins must be grabbed from the bank to guarantee grabbing at least:

(a) 3 pennies?

(b) 3 coins of the same kind?

(c) 3 quarters?

Solutions. (a) $\boxed{\text{24 coins}}$ There are $8 + 10 + 3 = 21$ non-pennies, and $12 + 21 = 33$ total coins. If we grab only 23 coins, they could be all 21 non-pennies and only 2 pennies. On the other hand, if we grab 24, then only $33 - 24 = 9$ coins remain in the bank, so we must have grabbed all but at most 9 of the pennies, i.e., we must have grabbed at least $12 - 9 = 3$ pennies.

(b) $\boxed{\text{9 coins}}$ There are $m = 4$ types of coins, so for the strong pigeonhole principle to guarantee n coins include at least $k = 3$ of the same type, we need (at least) $n = 1 + m(k - 1) = 9$.

(c) $\boxed{\text{33 coins}}$ If we grab only 32 coins, it is possible that the one remaining coin is a quarter, in which case we will not have all three. Conversely, if we grab all 33 coins, then in particular we will have grabbed all three quarters.

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3. Section 2.3, #7

For any set $S \subseteq \mathbb{Z}$ with $|S| = 3$, (i.e., for any set of three distinct integers), prove that S contains a pair whose sum is even (i.e., there exist *distinct* $m, n \in S$ such that $m + n$ is even).

Proof. Given such S thinking of "even" and "odd" as two pigeonholes, there are two different elements $m \neq n \in S$ such that x and y are either both odd or both even.

If they are both odd, then there are integers $a, b \in \mathbb{Z}$ such that $x = 2a + 1$ and $y = 2b + 1$, so $x + y = 2(a + b + 1)$ is even.

If they are both even, then there are integers $a, b \in \mathbb{Z}$ such that $x = 2a$ and $y = 2b$, so $x + y = 2(a + b)$ is even. QED

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4. Section 2.3, #9

Recall that the set $\mathbb{Z} \times \mathbb{Z}$ (of points in the plane both of whose coordinates are integers) is called the set of *lattice points in the plane*. For any five lattice points $(x_1, y_1), \ldots, (x_5, y_5) \in \mathbb{Z} \times \mathbb{Z}$, prove that there is a pair whose midpoint is also a lattice point.

Proof. There are four types of lattice points (x, y): those for which x and y are both even, those for which they are both odd, those for which x is odd and y is even, and those for which x is even and y is odd.

By the pigeonhole principle, there are two such points, WLOG (x_1, y_1) and (x_2, y_2), for which x_1 and x_2 have the same parity, and y_1 and y_2 have the same parity. By the same argument as in the previous problem, it follows that both $x_1 + x_2$ and $y_1 + y_2$ are even.

[Here's that argument again: we have $x_1 = 2a + i$ and $x_2 = 2b + i$ for integers a, b, i with i being either 0 or 1, so $x_1 + x_2 = 2(a + b + i)$ is even. Similarly for $y_1 + y_2$.]

So there are integers $m, n \in \mathbb{Z}$ such that $x_1 + x_2 = 2m$ and $y_1 + y_2 = 2n$. Thus, the midpoint $(\frac{x_1+x_2}{2}, \frac{y_1+y+2}{2}) = (m, n)$ is a lattice point. QED

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5. Section 3.1, Problems 3(b) and 4

(a) [#3b]: Let a, b, c, d be nonzero integers. Prove that if $a|b$ and $d|c$, then $(ad)|(bc)$

(b) [#4]: Prove, or disprove via counterexample: Let a, b, c be nonzero integers.

If $a|(bc)$, then $a|b$ or $a|c$.

Proofs. (a): Given $a, b, c, d \in \mathbb{Z}$ nonzero with $a|b$ and $d|c$, there exist integers $m, n \in \mathbb{Z}$ such that $b = ma$ and $c = nd$. Thus, $bc = (ma)(nd) = (mn)(ad)$. Since $mn \in \mathbb{Z}$, we have $(ad)|(bc)$. QED

(b): This statement is $\boxed{\text{false}}$

To prove this, let $a = 6$ and $b = 2$ and $c = 3$. Then $a|(bc)$, since $bc = 6 = 1 \cdot a$ is divisible by a, but neither $b = 2$ nor $c = 3$ is divisible by $a = 6$. QED

Note 1: There are *many* other counterexamples possible for part (b). The key is to pick a to be composite (i.e., $a \geq 2$ but not prime), with some of the factors of a dividing b, and the others dividing c.

Note 2: The answers and solutions to this problem are unchanged if we remove the restriction that these integers are all nonzero. I just included that restriction because the book did; but the problem is unchanged if we allow a, b, c, d to be *any* integers, including 0.

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6. Section 3.1, #12(a,b,e,h)

For each of the following pairs of integers a and b, find the integers $q, r \in \mathbb{Z}$ such that $b = qa + r$ and $0 \leq r < |a|$.

(a) $a = 73$, $b = 25$ (b) $a = 25$, $b = 73$ (e) $a = 79$, $b = -17$ (h) $a = 13$, $b = -37$

Solutions. (a): $\boxed{q = 0 \text{ and } r = 25}$ since we have $b = 25 = 0 \cdot 73 + 25$ with $0 \leq 25 < |73|$.

(b): $\boxed{q = 2 \text{ and } r = 23}$ since we have $b = 73 = 2 \cdot 25 + 23$ with $0 \leq 23 < |25|$.

(c): $\boxed{q = -1 \text{ and } r = 62}$ since we have $b = -17 = (-1) \cdot 79 + 62$ with $0 \leq 62 < |79|$.

(d): $\boxed{q = -3 \text{ and } r = 2}$ since we have $b = -37 = (-3) \cdot 13 + 2$ with $0 \leq 2 < |13|$.

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7. Section 3.1, #26

Prove that for every $n \in \mathbb{N}$, we have $6|(7^n - 1)$

Proof (Method 1). By induction on $n \geq 1$.

Base Case: $n = 1$. Then $7^n - 1 = 7 - 1 = 6$ is divisible by 6.

Inductive Step: For some particular $n \geq 1$, suppose that $6|(7^n - 1)$, so there is an integer $m \in \mathbb{Z}$ such that $7^n - 1 = 6m$. Then

$$7^{n+1} - 1 = (7^{n+1} - 7^n) + (7^n - 1) = 7^n \cdot (7 - 1) + 6m = 6(7^n + m).$$

Since $7^n + m \in \mathbb{Z}$, we have $6|(7^{n+1} - 1)$. QED

Proof (Method 2). Given $n \geq 1$, let $m = 1 + 7 + 7^2 + \cdots + 7^{n-1}$, which is an integer because $n - 1 \geq 0$, and hence all the terms here are of the form 7^i with $i \geq 0$, and hence they are all integers. Then

$$6m = (7 - 1)(1 + 7 + 7^2 + \cdots + 7^{n-1}) = 7 + 7^2 + \cdots + 7^n - 1 - 7 - \cdots - 7^{n-1} = 7^n - 1,$$

since for each $i = 1, \ldots, n-2$, the term 7^i appears once as $+7^i$ and once as -7^i; and the only other terms are 7^n and -1. QED