Solutions to Homework #7

1. Section 2.1, #22
For any rational number $x \in \mathbb{Q}$ and any irrational real number $y \in \mathbb{R} \smallsetminus \mathbb{Q}$, prove that $x + y$ is irrational.

Proof. Suppose (towards contradiction) that $x+y$ is rational. Then $y = (x+y) - x$ is a difference of two rational numbers and hence also rational, contradicting the hypotheses. Thus, $x + y$ must be irrational. QED

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2. Section 2.1, #7, variant
We say an integer $z \in \mathbb{Z}$ is an *additive identity element for* $\mathbb{Z}$ (or simply an *additive identity*, for short) if the following statement is true:

$$\forall n \in \mathbb{Z}, \quad \text{we have} \quad z + n = n.$$

We saw a proof in class that 0 is an additive identity element for $\mathbb{Z}$. Prove that it is unique.

[That is, prove that if $z \in \mathbb{Z}$ is an additive identity element, then we must have $z = 0$.]

Proof. Given $z_1, z_2 \in \mathbb{Z}$ that are both additive identities, we have that

$$z_2 = z_1 + z_2 = z_2 + z_1 = z_1,$$

where the first equality is because z_1 is an additive identity, the second is by commutativity, and the third is because z_2 is an additive identity. QED

Note: You may be asking, "Wait, what? How are we done already?" But remember, that's what the structure of a uniqueness proof is: the first line is, "Given two such things," and the last line is to conclude that they are equal. And we did that.

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3. Section 2.2, #2(b,h)
Prove the following by induction n.

(b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$

(g) $\dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!} < 3 - \dfrac{2}{(n+1)!}$ for all $n \in \mathbb{N} \smallsetminus \{1\}$

Proof. (b): By induction on $n \geq 1$:

Base Case: For $n = 1$, the left side is $1^3 = 1$, and the right side is $\dfrac{1^2(2)^2}{4} = 1$, so they are equal.

Inductive Step: Assume the statement is true for some $n \geq 1$. Then

$$1^3 + 2^3 + 3^3 + \cdots + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}\big[n^2 + 4(n+1)\big] = \frac{(n+1)^2(n+2)^2}{4}$$

QED

(g): By induction on $n \geq 2$:

Base Case: For $n = 2$, we have

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = 1 + 1 + \frac{1}{2} = 3 - \frac{1}{2} < 3 - \frac{2}{6} = 3 - \frac{1}{(2+1)!}.$$

Inductive Step: Assume the statement is true for some $n \geq 2$. Then

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n+1)!} < 3 - \frac{2}{(n+1)!} + \frac{1}{(n+1)!} = 3 - \frac{1}{(n+1)!} < 3 - \frac{2}{(n+2)!}$$

QED

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4. Section 2.2, #3

Find the value of $2^0 + 2^1 + 2^2 + \cdots + 2^n$ for $n = 0, 1, 2, 3, 4$. Make a conjecture about the value of the sum for $n \in \mathbb{N} \cup \{0\}$. Prove your conjecture.

Proof. We compute:

$$2^0 = 1, \quad 2^0 + 2^1 = 1 + 2 = 3, \quad 2^0 + 2^1 + 2^2 = 3 + 4 = 7$$

$$2^0 + 2^1 + 2^2 + 2^3 = 7 + 8 = 15, \quad \text{and} \quad 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 15 + 16 = 31.$$

We conjecture that $\boxed{2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \text{ for all } n \geq 0}$

We prove our conjecture by induction on $n \geq 0$.
Base Case: For $n = 0$, we have $2^0 = 1 = 2^1 - 1$.

Inductive Step: Assume the statement is true for some $n \geq 0$. Then

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = (2^{n+1} + 2^{n+1}) - 1 = 2^{n+2} - 1$$

QED

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5. Section 2.2, #9

Prove *Bernoulli's inequality*: For every $\alpha \in \mathbb{R}$ with $\alpha > -1$ and $\alpha \neq 0$, and for every $n \in \mathbb{N} \setminus \{1\}$, we have $(1 + \alpha)^n > 1 + n\alpha$.

Proof. Given $\alpha \in \mathbb{R}$ with $\alpha > -1$ and $\alpha \neq 0$, we proceed by induction on $n \geq 2$.

Base Case: For $n = 2$, we have $(1 + \alpha)^2 = 1 + 2\alpha + \alpha^2 < 1 + n\alpha$, because $\alpha \neq 0$ and hence $\alpha^2 > 0$.

Inductive Step: Assume the statement is true for some $n \geq 1$. Then because $1 + \alpha > 0$, we have

$$(1 + \alpha)^{n+1} = (1 + \alpha)^n (1 + \alpha) > (1 + n\alpha)(1 + \alpha) = 1 + (n + 1)\alpha + n\alpha^2 > 1 + (n + 1)\alpha$$

QED

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6. Section 2.2, #19 (with parts a, b, c)

Define a sequence of numbers $a_0, a_1, a_2, \ldots$ as follows:

$$a_0 = 0, \ a_1 = 1, \ \text{and for all } n \geq 2, \ \text{we have } a_n = 2a_{n-1} - a_{n-2} + 2.$$

(a) Find a_2, a_3, a_4, a_5

(b) Conjecture a formula for a_n for all $n \in \mathbb{N} \cup \{0\}$.

(c) Prove the conjectural formula you stated in part (b).

Proof. (a): We compute

$$a_2 = 2(1) - 0 + 2 = 4, \quad a_3 = 2(4) - 1 + 2 = 9, \quad a_4 = 2(9) - 4 + 2 = 16.$$

(b): We conjecture that $a_n = n^2$ for all $n \geq 0$.

(c): We proceed by induction on $n \geq 0$.
Base Cases: We have $a_0 = 0 = 0^2$ and $a_1 = 1 = 1^2$.

Inductive Step: Assume, for some $n \geq 1$, that the conjecture is true for both $n - 1$ and n. Then

$$a_{n+1} = 2a_n - a_{n-1} + 2 = 2n^2 - (n - 1)^2 + 2 = n^2 + 2n + 1 = (n + 1)^2$$

QED

Note: We needed to do two values of n in the base case, and also assume the previous two values in the inductive step, because the argument in the inductive step turned out to require knowing that the formula holds for both n and $n - 1$, and not just for n.