Solutions to Homework #6

1. Section 2.1, #9
Let $a, b, c \in \mathbb{R}$, and let $p(x)$ be the polynomial $p(x) = ax^2 + bx + c$. Prove that $p(1) = p(-1)$ if and only if $p(2) = p(-2)$.

Proof. ($\Rightarrow$) We have

$$a + b + c = p(1) = p(-1) = a - b + c,$$

and hence (by adding $-a + b - c$ to both sides) we have $2b = 0$, and therefore $b = 0$. Thus,

$$p(2) = 4a + 2b + c = 4a + c = 4a - 2b + c = p(-2).$$ QED ($\Rightarrow$)

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and hence (by adding $-4a + 2b - c$ to both sides) we have $4b = 0$, and therefore $b = 0$. Thus,

$$p(1) = a + b + c = a + c = a - b + c = p(-1).$$ QED

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2. Section 2.1, #12 Prove that for any integer $n \in \mathbb{Z}$, the number $n^3 + n$ is an even integer.

Proof. Given $n \in \mathbb{Z}$, we consider two cases.

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.
So $n^3 + n = 8k^3 + 2k = 2(4k^3 + k)$. Since $4k^3 + k \in \mathbb{Z}$ is an integer, we have that $n^3 + n$ is 2 times an integer, and hence is even.

Case 2: n is not even, so n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.
So $n^3+n = (2k+1)^3+(2k+1) = 8k^3+12k^2+6k+1+(2k+1) = 8k^3+12k^2+8k+2 = 2(4k^3+6k^2+4k+1)$.
Since $4k^3 + 6k^2 + 4k + 1 \in \mathbb{Z}$ is an integer, we have that $n^3 + n$ is 2 times an integer, and hence is even.
QED

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3. Section 2.1, #15(a)
Let $a \in \mathbb{Z}$ be an integer. Prove that a is a multiple of 3 if and only if a may be written as the sum of three consecutive integers.

Proof. Given $a \in \mathbb{Z}$ arbitrary:
($\Rightarrow$) Assuming a is a multiple of 3, there is an integer $k \in \mathbb{Z}$ such that $a = 3k$. Then $k - 1$ and k and $k + 1$ are three consecutive integers, and we have

$$(k - 1) + k + (k + 1) = 3k = a$$

($\Leftarrow$) Assuming a is the sum of three consecutive integers, call those integers x, $x + 1$, and $x + 2$, with $x \in \mathbb{Z}$. Let $k = x + 1 \in \mathbb{Z}$. Then

$$a = x + (x + 1) + (x + 2) = 3x + 3 = 3k,$$

so a is indeed a multiple of 3. QED

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4. Section 2.1, #17
Let $m, n \in \mathbb{Z}$ be integers. Prove that the following statements are equivalent:

(a) $m^2 - n^2$ is even. (b) $m - n$ is even. (c) $m^2 + n^2$ is even.

Proof. (a)$\Rightarrow$(b): Let's prove the contrapositive. Suppose that $m - n$ is not even, i.e., that $m - n$ is odd. Then there is an integer $k \in \mathbb{Z}$ such that $m - n = 2k + 1$, or equivalently, $m = n + 2k + 1$. Hence,

$$m^2 - n^2 = (n + 2k + 1)^2 - n^2 = n^2 + 4kn + 2n + 4k^2 + 4k + 1 - n^2 = 2(2kn + n + 2k^2 + 2k) + 1.$$

Since $2kn + n + 2k^2 + 2k \in \mathbb{Z}$ is an integer, we have that $m^2 - n^2$ is odd, completing our proof of the contrapositive. QED (a)$\Rightarrow$(b) (b)⇒(c): Since $m-n$ is even, there is an integer $k\in\mathbb{Z}$ such that $m-n=2k$, and hence that $m=2k+n$. Thus,

$$m^2+n^2=(2k+n)^2+n^2=4k^2+4kn+2n^2=2(2k^2+2kn+n^2).$$

Since $2k^2+2kn+n^2\in\mathbb{Z}$ is an integer, we have that m^2+n^2 is even. QED (b)⇒(c)

(c)⇒(a): Since m^2+n^2 is even, there is an integer $k\in\mathbb{Z}$ such that $m^2+n^2=2k$. Thus,

$$m^2-n^2=(m^2+n^2)-2n^2=2k-2n^2=2(k-n^2).$$

Since $k-n^2\in\mathbb{Z}$ is an integer, we have that m^2-n^2 is even. QED (c)⇒(b)

QED

Note: There are lots of other ways to do this. For example, here's an alternative proof that (a)⇒(b): Since m^2-n^2 is even, there is an integer $k\in\mathbb{Z}$ such that $m^2-n^2=2k$. Thus,

$$2k=m^2-n^2=(m-n)(m+n).$$

Since 2 is prime, this equality implies that either $m-n$ is divisible by 2, or that $m+n$ is divisible by 2. We consider these two cases separately.

Case 1. $m-n$ is divisible by 2. Then $m-n$ is even, as desired.

Case 2. $m+n$ is divisible by 2. Then there is an integer $\ell\in\mathbb{Z}$ such that $m+n=2\ell$. Hence,

$$m-n=(m+n)-2n=2\ell-2n=2(\ell-n).$$

Since $\ell-n\in\mathbb{Z}$ is an integer, we have that $m-n$ is even, as desired. QED (a)⇒(b)

Note, continued: And here's a proof going in the opposite direction of implications:
(b) ⇒ (a): Since $m-n$ is even, there exists $k\in\mathbb{Z}$ such that $m-n=2k$.
So $m^2-n^2=(m-n)(m+n)=2k(m+n)$.
Since $k(m+n)\in\mathbb{Z}$, we have that m^2-n^2 is even. QED (b)⇒(a)

(a)⇒(c): Since m^2-n^2 is even, there exists $k\in\mathbb{Z}$ such that $m^2-n^2=2k$.
So $m^2+n^2=m^2-n^2+2n^2=2k+2n^2=2(k+n^2)$.
Since $k+n^2\in\mathbb{Z}$, we have that m^2+n^2 is even. QED (a)⇒(c)

(c)⇒(b): Since m^2+n^2 is even, there exists $k\in\mathbb{Z}$ such that $m^2+n^2=2k$.
So $(m-n)^2=m^2-2mn+n^2=2k-2mn=2(k-mn)$.
Since $k-mn\in\mathbb{Z}$, we have that $(m-n)^2$ is even. But then, since $(m-n)^2$ is even, we have by Theorem 2.1.9(b) that $m-n$ is even. QED (c)⇒(a)

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5. Section 2.1, #5
Let $A,B,C\in\mathbb{Z}$ be integers with $A,B\neq 0$, and let L be the line $\{(x,y)\in\mathbb{R}^2 \,|\, Ax+By=C\}$. Suppose that L contains a lattice point. Prove that L contains infinitely many lattice points.

Proof. The hypotheses state that there are integers $x_0,y_0\in\mathbb{Z}$ such that the point (x_0,y_0) lies on L. In particular, we have $Ax_0+By_0=C$.

For each integer $n\in\mathbb{Z}$, let $x_n=x_0+Bn\in\mathbb{Z}$, and let $y_n=y_0-An\in\mathbb{Z}$. So (x_n,y_n) is a lattice point.

In addition, for any two integers $m,n\in\mathbb{Z}$, if $(x_m,y_m)=(x_n,y_n)$ are the same point, then

$$x_0+Bm=x_0+Bn \quad\text{and}\quad y_0-Am=y_0-An,$$

and hence $B(m-n)=0$ and $A(m-n)=0$. Since at least one of A,B is nonzero, it follows that $m-n=0$, and hence $m=n$. Thus, we have produced infinitely many lattice points $\{(x_n,y_n)\,|\,n\in\mathbb{Z}\}$.

Finally, for each $n\in\mathbb{Z}$, we have

$$Ax_n+By_n=A(x_0+Bn)+B(y_0-An)=Ax_0+By_0+ABn-ABn=C+0=C,$$

so that each (x_n, y_n) is indeed a point on the line L. QED

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6. Section 2.1, #8(a)
For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x. Prove that for all $x, y \in \mathbb{R}$, we have $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. Given any $x, y \in \mathbb{R}$, let $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$ so that $m, n \in \mathbb{Z}$ with $m \leq x < m + 1$ and $n \leq y < n + 1$.

So $m + n$ is an integer, and $m + n \leq x + y$. Thus, $\lfloor x + y \rfloor \geq m + n$, since $\lfloor x + y \rfloor$ is the *largest* integer that is less than or equal to $x + y$, and $m + n$ is *an* integer that is less than or equal to $x + y$. Hence,

$$\lfloor x \rfloor + \lfloor y \rfloor = m + n \leq \lfloor x + y \rfloor,$$

proving the first desired inequality.

In addition, we have

$$\lfloor x + y \rfloor \leq x + y < (m + 1) + (n + 1) = m + n + 2.$$

Since the integer $\lfloor x + y \rfloor$ is strictly less than the integer $m + n + 2$, we must have

$$\lfloor x + y \rfloor \leq (m + n + 2) - 1 = m + n + 1 = \lfloor x \rfloor + \lfloor y \rfloor + 1,$$

proving the second desired inequality. QED