Solutions to Homework #5

1. Section 1.6, Problem 2 (with parts a–c) (12 points)
Consider the following statements:

I = Interest rates go down.
H = More people buy houses.
S = The stock market goes up.
U = Unemployment goes up.

Write each of the following implications, and its converse, and its contrapositive, in words:

(a) $I \Rightarrow S$
(b) $(\sim U) \Rightarrow H$
(c) $S \Rightarrow (I \wedge H)$

Solution. (a) Original: If interest rates go down, then the stock market goes up

Converse: If the stock market goes up, then interest rates go down

Contrapositive: If the stock market does not go up, then interest rates do not go down

(b) Original: If unemployment does not go up, then more people buy houses

Converse: If more people buy houses, then unemployment does not go up

Contrapositive: If more people do not buy houses, then unemployment goes up

(c) Original: If the stock market goes up, then interest rates go down and more people buy houses

Converse: If interest rates go down and more people buy houses, then the stock market goes up

Contrapositive:
If interest rates do not go down or more people do not buy houses, then the stock market does not go up

Note: Really, the negation $\sim H$ of "more people buy houses" is technically *not* "more people do not buy houses" as I said in the contrapositives for (b) and (c). After all, that would mean that *more* people are not buying houses, whereas $\sim H$ is really "it is not true that more people buy houses," which includes the possibility that the number of people buying houses remains the same.
However, given how contorted the truly correct language "it is not true that more people buy houses" would make the contrapositives in (b) and (c), I'm stating the simpler language "more people do not buy houses" and will accept either version on the homework.
On the other hand, the negation $\sim I$ is *not* "interest rates go up" but rather "interests rates do not go down," which includes the possibility that interest rates do not change. Similarly for the negations $\sim S$ and $\sim U$.

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2. Section 1.6, Problem 4 (a–d) (12 points)
Using the statements I, H, S, U above, write each of the following statements in symbolic form.

(a) Interest rates go down only if unemployment goes up.
(b) The stock market goes up if more people buy houses.
(c) Unemployment does not go up if interest rates go down and more people buy houses.
(d) If interest rates do not go down, the stock market does not go up.

Solution. (a) $I \Rightarrow U$

(b) $H \Rightarrow S$

(c) $(I \wedge H) \Rightarrow (\sim U)$

(d) $(\sim I) \Rightarrow (\sim S)$

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3. Section 1.6, Problem 7(a,b,d) (18 points)

Assume all variables in this problem represent real numbers. For each of the following implication statements, state its converse. In addition, determine the truth value of each implication and its converse, and briefly explain why.

(a) $x^2 = 4$ only if $x = 2$.

(b) If $2x \leq x$, then $x^2 > 0$.

(d) If x is an integer, then $\sqrt{x}$ is an integer.

Solution. (a) Converse: $\boxed{\text{If } x = 2, \text{ then } x^2 = 4}$ or if you prefer, $\boxed{x = 2 \text{ only if } x^2 = 4}$

$\boxed{\text{The original is } \textbf{false}}$ because $x = -2$ makes $x^2 = 4$ true but makes $x = 2$ false.

$\boxed{\text{The converse is } \textbf{true}}$ because $x = 2$ makes $x^2 = 4$ true.

(b) Converse: $\boxed{\text{If } x^2 > 0, \text{ then } 2x \leq x}$

$\boxed{\text{The original is } \textbf{false}}$ because $x = 0$ makes $2x \leq x$ true but makes $x^2 > 0$ false.

$\boxed{\text{The converse is } \textbf{false}}$ because $x = 1$ makes $x^2 > 0$ true but $2x \leq x$ false.

(d) Converse: $\boxed{\text{If } \sqrt{x} \text{ is an integer, then } x \text{ is an integer}}$

$\boxed{\text{The original is } \textbf{false}}$ because $x = 2$ is an integer but $\sqrt{2}$ is not.

$\boxed{\text{The converse is } \textbf{true}}$ because if $\sqrt{x}$ is an integer, then $x = (\sqrt{x})^2$ is the square of an integer and hence also an integer.

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4. Section 1.6, Problem 8 (with parts a–c) (18 points)

In all parts below, make sure to (briefly) justify your answer, as always.

(a) Find a condition on $x \in \mathbb{R}$ which is necessary and sufficient for $x^2 \in [4, 9]$

(b) Find a condition on $x \in \mathbb{R}$ which is necessary but not sufficient for $x^2 \in [4, 9]$.

(c) Find a condition on $x \in \mathbb{R}$ which is sufficient but not necessary for $x^2 \in [4, 9]$.

Solution. (a) The condition that $\boxed{|x| \in [2, 3]}$, i.e., that $x \in [-3, -2] \cup [2, 3]$, is necessary and sufficient.

It is sufficient, because if $|x| \in 2, 3$, then $2 \leq x \leq 3$, so $2^2 \leq x^2 \leq 9$.

It is necessary, because if $x^2 \in [4, 9]$, then $4 \leq x^2 \leq 9$, so $2 \leq |x| \leq 3$.

(b) The condition $\boxed{x \in [-3, 3]}$ i.e., that $|x| \leq 3$, is necessary but not sufficient.

It is necessary, because if $x^2 \in [4, 9]$, then $x^2 \leq 9$, so $|x| \leq 3$.

It is *not* sufficient, because $x = 1$ satisfies the condition $x \in [-3, 3]$, but $x^2 = 1 \notin [4, 9]$.

(c) The condition $\boxed{x = 3}$ is sufficient but not necessary.

It is sufficient, because if $x = 3$, then $x^2 = 9 \in [4, 9]$.

It is *not* necessary, because $x = 2$ satisfies the condition $x^2 \in [4, 9]$, but $x \neq 3$.

Note: There are a **LOT** of other possible correct answers for parts (b) and (c).

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5. (12 points)

Let A be a set, and suppose $R \subseteq A \times A$.

(a) Find the contrapositive of the implication $\big[(a, b) \in R \wedge (b, a) \in R\big] \Rightarrow a = b$

(b) Find the negation of the statement $\forall a, b \in A$, $\big[(a, b) \in R \wedge (b, a) \in R\big] \Rightarrow a = b$

Solution. (a) The contrapositive is:

$$a \neq b \Rightarrow \big[(a, b) \notin R \vee (b, a) \notin R\big].$$

(b) The negation is:

$$\exists a, b \in A \text{ s.t. } \big[(a, b) \in R \wedge (b, a) \in R\big] \wedge a \neq b.$$