Solutions to Homework #4

1. Section 1.4, Problem 7(a) (5 points) Verify the equation $\big(\sim(P \wedge Q)\big) \wedge (\sim Q) = \sim Q$ using a truth table.

Solution. Here is the truth table:

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim Q$	$\big(\sim(P \wedge Q)\big) \wedge (\sim Q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	T	T	T

The last two columns verify the desired equation.

---

2. Section 1.4, Problem 12, slight variant (7 points)
Construct a truth table with columns for $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$.
Briefly explain why it would be bad to use no parentheses when writing $P \vee Q \wedge R$.

Solution. Here is the truth table:

P	Q	R	$(P \vee Q)$	$(P \vee Q) \wedge R$	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

The 5th and 7th (i.e., last and third-to-last) columns are different, for example in the second row. [I.e., when P and Q are true but R is false. They are also different in the fourth row.]
So the two expressions are different. Thus, if we didn't use parentheses, it would be impossible to tell which of the two expressions was meant by $P \vee Q \wedge R$.

---

3. Section 1.5, Problem 2(a,c,d) (12 points)
Write each sentence below as a mathematical statement using $\forall$ and/or $\exists$.

(a) For every positive real number ε, there is a natural number n with $\frac{1}{n} < \varepsilon$

(c) For every positive real number ε, there is a positive real number δ such that $x^2 < \varepsilon$ whenever x is a real number with $|x| < \delta$.

(d) There exists an integer m with the property that for every integer x, there exists an integer y with $xy = m$.

Solution. (a) $\boxed{\forall \varepsilon \in \mathbb{R} \text{ s.t. } \varepsilon > 0, \exists n \in \mathbb{N} \text{ with } \frac{1}{n} < \varepsilon}$

(c) $\boxed{\forall \varepsilon \in \mathbb{R} \text{ s.t. } \varepsilon > 0, \exists \delta \in \mathbb{R} \text{ with } \delta > 0 \text{ s.t. } \forall x \in \mathbb{R} \text{ with } |x| < \delta, \text{ we have } x^2 < \varepsilon.}$

(d) $\boxed{\exists m \in \mathbb{Z} \text{ s.t. } \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ with } xy = m}$

Note: This problem was just asking you to translate sentences into mathematical shorthand, and *not* asking you about whether or not the statements are true or false. In fact, statements (a) and (c) are true, and (d) is false. Can you see why?

---

4. Section 1.5, Problem 5(a,c,e) (15 points)
For each statement below, determine whether it is true or false (and briefly explain why), and give the negation of the statement.

(a) $\forall x \in \mathbb{R}$, $\exists a \in \mathbb{R}$ with $|x| < a$
(c) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t. $xy = 1$.
(e) $\forall a \in \mathbb{R}$, $\sqrt{a^2} = a$.

Solution. (a) **True**, because for any $x \in \mathbb{R}$, we can define $a = |x| + 1$. Then $a \in \mathbb{R}$, and $|x| < a$.

The negation is: $\boxed{\exists x \in \mathbb{R} \text{ s.t. } \forall a \in \mathbb{R}, \text{ we have } |x| \geq a}$

(c) **False**, because for $x = 0$, we have $x \in \mathbb{R}$, but there is no $y \in \mathbb{R}$ such that $0 \cdot y = 1$.

The negation is: $\boxed{\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, \text{ we have } xy \neq 1}$

(e) **False**, because for $a = -1$, we have $a \in \mathbb{R}$, but $\sqrt{a^2} = \sqrt{1} = 1 \neq -1$.

The negation is: $\boxed{\exists a \in \mathbb{R} \text{ s.t. } \sqrt{a^2} \neq a}$

Note on (e): there are a *lot* of counterexamples; choosing a to be any negative number will make $\sqrt{a^2} \neq a$. However, to *prove* that the statement is false, we merely need to produce *one* counterexample. Indeed, stating the negation — that **there exists** $a \in \mathbb{R}$ such that blah blah blah — illustrates that we only need to find one.

---

5. Section 1.5, Problem 6(a,c,d) (15 points)
For each statement below, determine whether it is true or false (and briefly explain why), and give the negation of the statement.
(a) $\exists x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}$, $\frac{y}{x} \in \mathbb{Z}$
(c) $\forall u \in \mathbb{N}$, $\exists v \in \mathbb{N} \smallsetminus \{u\}$ s.t. $\frac{v}{u} \in \mathbb{N}$
(d) $\forall u \in \mathbb{N}$, $\exists v \in \mathbb{N} \smallsetminus \{u\}$ s.t. $\frac{u}{v} \in \mathbb{N}$

Solution. (a) **True**, because $x = 1 \in \mathbb{Z}$ has the property that for all $y \in \mathbb{Z}$, we have $\frac{y}{x} = y \in \mathbb{Z}$.

The negation is: $\boxed{\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ s.t. } \frac{y}{x} \notin \mathbb{Z}}$

Note: to verify that the original statement is true, we could also have selected $x = -1$. But again, we don't need to find *all* such x to prove a "there exists" statement.

(c) **True**, because for any $u \in \mathbb{N}$, we can choose $v = 2u \in \mathbb{N} \smallsetminus \{u\}$, which has $\frac{v}{u} = 2 \in \mathbb{N}$.

The negation is: $\boxed{\exists u \in \mathbb{N} \text{ s.t. } \forall v \in \mathbb{N} \smallsetminus \{u\}, \text{ we have } \frac{v}{u} \notin \mathbb{N}}$

(d) **False**, because if we choose $u = 1 \in \mathbb{N}$, then for any $v\mathbb{N} \smallsetminus \{1\}$, we have $v \geq 2$, so $\frac{u}{v} = \frac{1}{v} \notin \mathbb{N}$.

The negation is: $\boxed{\exists u \in \mathbb{N} \text{ s.t. } \forall v \in \mathbb{N} \smallsetminus \{u\}, \text{ we have } \frac{u}{v} \notin \mathbb{N}}$

---

6. Section 1.5, Problem 8(c) (8 points)
For the following statement, determine whether it is true or false (and briefly explain why), and give the negation of the statement:

$$\forall k \in \mathbb{N}, \exists S \in \mathcal{P}\big(\{1, 2, \ldots, k\}\big) \text{ s.t. } S \neq \varnothing \text{ and } \forall x, y \in S, x - y \text{ is even.}$$

Solution. **True**. Here's a proof:
Given $k \in \mathbb{N}$, let $S = \{1\} \in \mathcal{P}\big(\{1, 2, \ldots, k\}\big)$.
Then $S \neq \varnothing$. Finally, to prove the "for all" statement at the end:
Given $x, y \in S$, we have $x = y = 1$ (since 1 is the only element of S), so $x - y = 0$ is even. QED

The negation is: $\boxed{\exists k \in \mathbb{N} \text{ s.t. } \forall S \in \mathcal{P}\big(\{1, 2, \ldots, k\}\big) \text{ s.t. } S \neq \varnothing, \exists x, y \in S \text{ s.t. } x - y \text{ is not even.}}$