Solutions to Homework #3

1. Section 1.2, Problem 15 (12 points)
For any sets A, B, C, prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. ($\subseteq$) Given $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$.
We have $x \in A$ and $y \in B$, so $(x, y) \in A \times B$.
We also have $x \in A$ and $y \in C$, so $(x, y) \in A \times C$.
Thus, $(x, y) \in (A \times B) \cap (A \times C)$. QED ($\subseteq$)

($\supseteq$) Given $(x, y) \in (A \times B) \cap (A \times C)$, we have $(x, y) \in A \times B$ and $(x, y) \in A \times C$.
Thus, $x \in A$ and $y \in B$ and $y \in C$. Hence, we have $x \in A$ and $y \in B \cap C$.
That is, $(x, y) \in A \times (B \cap C)$. QED ($\supseteq$) QED

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2. Section 1.2, Problem 20 (6 points)
At a diner, the special consists of your choice of meatloaf or chicken, served with mashed potatoes and your choice of one vegetable from seven on the menu and one dessert from five on the menu. How many ways can you place an order for the special?

(*Note*: Please approach this problem by considering the set of possible orders as $E \times V \times D$, where E is the set of possible entrees, V is the set of possible vegetables, and D is the set of possible desserts.)

Solution. There are $\boxed{\text{70 possible orders}}$

Let $E = \{\text{meatloaf}, \text{chicken}\}$ be the set of entree options, let V be the set of vegetable options, and let D be the set of dessert options.
We have $|E| = 2$ and $|V| = 7$ and $|D| = 5$.
Each possible order looks like (e, v, d), where $e \in E$ and $v \in V$ and $d \in D$, so the set of possible orders is $E \times V \times D$.
By a theorem, then, the number of possible orders is $|E \times V \times D| = |E| \cdot |V| \cdot |D| = 2 \cdot 7 \cdot 5 = 70$.

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3. Section 1.3, Problem 6 (15 points)
For each $m \in \mathbb{N}$, let $C_m = \{x \in \mathbb{R} | m - 1 \leq x^2 < m\}$. Let $\mathcal{C} = \{C_m | m \in \mathbb{N}\}$. Is $\mathcal{C}$ a partition of $\mathbb{R}$?
Answer yes or no, and then prove your answer.

Solution. $\boxed{\text{Yes, } \mathcal{C} \text{ is a partition of } \mathbb{R}}$

Proof. (Each nonempty): For each $m \in \mathbb{N}$, we must show that $C_m \neq \varnothing$.
Let $x = \sqrt{m-1}$, which is indeed a real number because $m - 1 \geq 0$. Thus, $x \in \mathbb{R}$ and $m - 1 = x^2$, so $x \in C_m$.

(Mutually disjoint): Given any $m, n \in \mathbb{N}$ distinct (i.e., $m \neq n$), without loss of generality, we may assume that $m < n$. For any $x \in C_m$, we have

$$x^2 < m - 1 \leq n, \quad \text{and hence} \quad x \notin C_n.$$

Thus, $C_m \cap C_n = \varnothing$.

(Union): We must show $\displaystyle\bigcup_{m \in \mathbb{N}} C_m = \mathbb{R}$.

($\subseteq$): We have $C_m \subseteq \mathbb{R}$ for every $m \in \mathbb{N}$ by definition of C_m. Thus, the union $\displaystyle\bigcup_{m \in \mathbb{N}} C_m$ is contained in $\mathbb{R}$.

($\supseteq$): Given any $x \in \mathbb{R}$, we have $x^2 \in [0, \infty)$. Let n be the greatest integer that is less than or equal to x^2, so that $n \leq x^2 < n + 1$. Since $x \geq 0$, we have $n \geq 0$. Let $m = n + 1$, so that $m \geq 1$ is an integer, and hence $m \in \mathbb{N}$. Then we have $x \in C_m$, so $x \in \displaystyle\bigcup_{m \in \mathbb{N}} C_m$. QED

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4. Section 1.3, Problem 9(a) (15 points)
For each $b \in \mathbb{R}$, let $I_b = \{(x, y) \in \mathbb{R}^2 | y = b\}$. Let $\mathcal{C} = \{I_b | b \in \mathbb{R}\}$. Is $\mathcal{C}$ a partition of $\mathbb{R} \times \mathbb{R}$?

Answer yes or no, and then prove your answer.

Solution. $\boxed{\text{Yes, } \mathcal{C} \text{ is a partition of } \mathbb{R}^2}$

Proof. (Each nonempty): For each $b \in \mathbb{R}$, we have $(0, b) \in I_b$, so $I_b \neq \varnothing$.

(Mutually disjoint): Given any $b, c \in \mathbb{R}$ distinct, and given any $(x, y) \in I_b$, we have $y = b \neq c$. Therefore, $(x, y) \notin I_c$. Thus, $I_b \cap I_c = \varnothing$.

(Union): We must show $\displaystyle\bigcup_{b \in \mathbb{R}} I_b = \mathbb{R}^2$.

($\subseteq$): We have $I_b \subseteq \mathbb{R}^2$ for every $b \in \mathbb{R}$ by definition of I_b. Thus, the union $\displaystyle\bigcup_{b \in \mathbb{R}} I_b$ is contained in $\mathbb{R}^2$.

($\supseteq$): Given any $(x, y) \in \mathbb{R}^2$, let $b = y \in \mathbb{R}$. Then $(x, y) \in I_b \subseteq \displaystyle\bigcup_{b \in \mathbb{R}} I_b$. QED

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5. Section 1.4, Problem 1(a,c,e,f) (12 points) Consider the following statements:

$S =$ Susan studies
$G =$ Susan gets good grades
$H =$ Susan gets help when needed

Write the following sentences symbolically:

(a) Susan studies but does not get good grades.
(c) It is not true that Susan studies and gets good grades.
(e) Susan studies or does not study, and she gets good grades.
(f) Susan studies, gets help when needed, and gets good grades.

Solutions. (a): $\boxed{S \wedge (\sim G)}$

(c): $\boxed{\sim (S \wedge G)}$

(e): $\boxed{(S \vee (\sim S)) \wedge G}$

(f): $\boxed{S \wedge H \wedge G}$

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6. Section 1.4, Problem 2(b,c,e,g) (12 points) Consider the following statements:

$P =$ Presidential candidates must be 35 years of age or older
$Q =$ Presidential candidates must be citizens of the United States
$R =$ Presidential candidates must have \$27 million

Write the following statements in words (as complete English sentences):

(b) $P \wedge Q$
(c) $Q \vee R$
(e) $(P \wedge Q) \vee R$
(g) $P \wedge (\sim R)$

Solution. (b): Presidential candidates must be 35 years of age or older, and citizens of the United States.

(c): Presidential candidates must be citizens of the United States, or must have \$27 million.

(e): Presidential candidates must be 35 years of age or older and citizens of the United States, or else they must have \$27 million.

(g): Presidential candidates must be 35 years of age or older, and they do not need to have \$27 million.

Note on (g): The negation of "they must have \$27 million" is *not* "'they must *not* have \$27 million," but rather that it is not true that they *must* have \$27 million. That is, the negation of "they must have \$27 million" is that they are not *required* to have \$27 million; but they are still *allowed* to have \$27 million.

Also note: I'm making no comment on whether any of the above statements are true or false; I'm simply explaining what, precisely, each one means.