Solutions to Homework #20

1. Section 6.3, #3(b): Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $|[a,b]| = |[0,1]|$

Proof. Let $m = b - a > 0$. Define $f : [0,1] \to [a,b]$ by $f(x) = a + mx$. Then f is indeed a function, because for any $x \in [0,1]$, we have $0 \le x \le 1$, so since $m > 0$, we have

$$a \le a + mx \le a + m = a + (b-a) = b,$$

so $f(x) \in [a,b]$, as desired.

1-1: Given $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$, we have $a + mx_1 = a + mx_2$, so $mx_1 = mx_2$, so $x_1 = x_2$ QED 1-1

Onto: Given $y \in [a,b]$, let $x = \frac{1}{m}(y-a)$, which is defined and real since $m \neq 0$. In fact, since $m > 0$ and $a \le y \le b$, we have

$$0 \le y - a \le b - a = m, \quad \text{so} \quad 0 \le \frac{1}{m}(y-a) \le 1,$$

and hence $x \in [0,1]$. Finally, we have $f(x) = a + m\left(\frac{1}{m}(y-a)\right) = a + (y-a) = y$. QED onto

QED

Note: After defining f as above, alternatively we could define its inverse function $g : [a,b] \to [0,1]$ by $g(y) = \dfrac{1}{m}(y-a)$.

We'd have to prove that g is actually a function into $[0,1]$ (very similar to in the onto proof above), and verify both compositions. (One is very similar to the rest of the onto proof, and the other has some similarities to the 1-1 proof.)

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2. Section 6.3, #3(c), variant: Prove that $|[0,\infty)| = |(0,1]|$

Proof. Define $f : [0,\infty) \to (0,1]$ by $f(x) = \dfrac{1}{x+1}$. Then f is indeed a function, because for any $x \in [0,\infty)$, we have $x \neq -1$, so $f(x) \in \mathbb{R}$ is defined, and we furthermore have $x \ge 0$, so

$$\frac{1}{x+1} \le \frac{1}{1} = 1, \quad \text{and} \quad \frac{1}{x+1} > 0.$$

Thus, $f(x) \in (0,1]$, as claimed. QED Function

1-1: Given $x_1, x_2 \in [0,\infty)$ such that $f(x_1) = f(x_2)$, we have

$$\frac{1}{x_1+1} = f(x_1) = f(x_2) = \frac{1}{x_2+1}, \quad \text{so} \quad x_1 + 1 = x_2 + 1, \quad \text{so} \quad x_1 = x_2.$$

QED 1-1

Onto: Given $y \in (0,1]$, let $x = \dfrac{1}{y} - 1$. Then $x \in \mathbb{R}$ is defined because $y \neq 0$. Furthermore, because $0 < y \le 1$, we have

$$\frac{1}{y} \ge 1, \quad \text{and hence} \quad x = \frac{1}{y} - 1 \ge 0.$$

Thus, $x \in [0,\infty)$. Finally, we have $f(x) = \dfrac{1}{(\frac{1}{y} - 1) + 1} = \dfrac{1}{1/y} = y$. QED Onto

QED

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3. Define $f : [0,\infty) \to (0,\infty)$ by $f(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z}, \\ x & \text{if } x \notin \mathbb{Z}. \end{cases}$

Prove that f is a function, and that it is bijective. [And therefore, that $|[0,\infty)| = |(0,\infty)|$.]

Proof. **Function**: Given $x \in [0,\infty)$, there are two cases.

First, if $x \in \mathbb{Z}$, then because $x \ge 0$, we have $f(x) = x + 1 \ge 1 > 0$, so $f(x) \in (0,\infty)$.

Second, if $x \notin \mathbb{Z}$, then we have $x \neq 0$, so because $x \ge 0$, we have $x > 0$. Thus, $f(x) = x \in (0,\infty)$.

QED Function

1-1: Given $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$, we consider two cases.
First, if $f(x_1) \in \mathbb{Z}$, then we must have $x_1 \in \mathbb{Z}$, or else we would have had $f(x_1) = x_1 \notin \mathbb{Z}$, a contradiction. Similarly, we must also have $x_2 \in \mathbb{Z}$. Thus,

$$x_1 + 1 = f(x_1) = f(x_2) = x_2 + 1 \quad \text{so} \quad x_1 = x_2.$$

Second, if $f(x_1) \notin \mathbb{Z}$, then we must have $x_1 \notin \mathbb{Z}$, or else we would have had $f(x_1) = x_1 + 1 \in \mathbb{Z}$, a contradiction. Similarly, we must also have $x_2 \notin \mathbb{Z}$. Thus,

$$x_1 = f(x_1) = f(x_2) = x_2 \quad \text{so} \quad x_1 = x_2.$$

QED 1-1

Onto: Given $y \in (0, \infty)$, we consider two cases:

First, if $y \in \mathbb{Z}$, then since $y > 0$, we in fact have $y \geq 1$. Let $x = y - 1 \geq 0$, so $x \in [0, \infty)$. Note also that $x \in \mathbb{Z}$, so $f(x) = x + 1 = y$, as desired.

Second, otherwise we have $y \notin \mathbb{Z}$. Define $x = y > 0$, so $x \in [0, \infty)$. Since $x \notin \mathbb{Z}$, we have $f(x) = x = y$, as desired. QED Onto

QED

Note: We could alternatively do this by defining the inverse map
$g : (0, \infty) \to [0, \infty)$ by $g(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Z}, \\ x & \text{if } x \notin \mathbb{Z}. \end{cases}$
and proving that it is indeed a function, and inverse to f.

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4. Section 6.3, #3(c), another variant: Prove that $|\mathbb{R}| = |(0, 1]|$

Proof. By Problem 3 of Homework 19, there is a bijective function $F_1 : \mathbb{R} \to (0, \infty)$.
By Problem 3 of this assignment, there is a bijective function $F_2 : (0, \infty) \to [0, \infty)$.
By Problem 2 of this assignment, there is a bijective function $F_3 : [0, \infty) \to (0, 1]$.

Define $G : \mathbb{R} \to (0, 1]$ by $G = F_3 \circ F_2 \circ F_1$.
Then G is indeed a function because the domains and target sets line up properly. And because each F_i is bijective, their composition G is also bijective.

So since we have a bijective function from $\mathbb{R}$ to $(0, 1]$, we have $|\mathbb{R}| = |(0, 1]|$. QED

Optional Challenge A. Section 6.3, #8: Let A be a set, and let $\mathcal{H} = \{f : A \to \{0, 1\}\}$. That is, $\mathcal{H}$ is the set of all functions from A to $\{0, 1\}$. Prove that $|\mathcal{P}(A)| = |\mathcal{H}|$.

Proof. For any $S \in \mathcal{P}(A)$, that is, for any subset $S \subseteq A$, define a function $f_S : A \to \{0, 1\}$ by

$$f_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Define $F : \mathcal{P}(A) \to \mathcal{H}$ by $F(S) = f_S$. Then F is indeed a function, because for each $S \in \mathcal{P}(A)$, we have that f_S is a function from A to $\{0, 1\}$, and hence $f_S \in \mathcal{H}$. It remains to show that F is bijective.

(**1-1**): Given $S, T \in \mathcal{P}(A)$ such that $F(S) = F(T)$, we have $f_S = f_T$, and we claim that $S = T$, which we now prove:

($\subseteq$): Given $x \in S$, we have $f_T(x) = f_S(x) = 1$, and hence $x \in T$.

($\supseteq$): Given $x \in T$, we have $f_S(x) = f_T(x) = 1$, and hence $x \in S$.

Thus, we indeed have $S = T$. QED 1-1

(**Onto**): Given $g \in \mathcal{H}$, i.e., given a function $g : A \to \{0, 1\}$, define $S = g^{-1}(1)$. Then $S \subseteq A$, so $S \in \mathcal{A}$. Hence $F(S) = f_S$ is a function from A to $\{0, 1\}$. We must show that $f_S = g$. These two functions both have the same target set $\{0, 1\}$ and same domain A, so we must only show that they have the same values.

Given $x \in A$, there are two cases:

Case 1: $g(x) = 1$. Then we have $x \in g^{-1}(1) = S$, so $f_S(x) = 1 = g(x)$.

Case 2: $g(x) \neq 1$. Then $g(x) = 0$, and we also have $x \notin g^{-1}(1) = S$, so $f_S(x) = 0 = g(x)$.

Thus, for all $x \in A$, we have $f_S(x) = g(x)$. So $f_S = g$. QED onto

QED

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Note: Alternatively, we could have defined a function $G : \mathcal{H} \to \mathcal{P}(A)$ by the following rule: for any $g \in \mathcal{H}$, i.e., for any function $g : A \to \{0,1\}$, define $G(g) = g^{-1}(1)$, i.e., $G(g) = \{x \in A \,|\, g(x) = 1\} \subseteq A$. Then we could either prove that G is 1-1 and onto (by a similar proof as above), or else show that G is the inverse of the function F in the proof above.

Within any of these variants, we could also swap the roles of 0 and 1 as long as we were consistent about maintaining that swap throughout the proof.

Optional Challenge B. Section 6.3, #6, slight variant:

Let $A = \{f : \{0,1\} \to \mathbb{N}\}$ be the set of all functions from $\{0,1\}$ to $\mathbb{N}$. Prove that A is countable.

Proof. Define a function $G : A \to \mathbb{N} \times \mathbb{N}$ by $G(f) = (f(0), f(1))$.

We claim that G is a bijective function. If we assume that claim for a moment, recall that by Theorem 6.3.9, there is a bijective function $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Thus, $H \circ G : A \to \mathbb{N}$ would also be a bijective function, and hence $|A| = |\mathbb{N}|$, as desired.

That is, it suffices to prove our claim, that G is a bijective function.

Function: Given any $f \in A$, we have $f(0), f(1) \in \mathbb{N}$, by definition of A.
Thus, $G(f) = (f(0), f(1)) \in \mathbb{N} \times \mathbb{N}$. QED Function

1-1: Given $f, g \in A$ such that $G(f) = G(g)$, we have that f and g are both functions from $\{0,1\}$ to $\mathbb{N}$, and that $(f(0), f(1)) = (g(0), g(1))$. Thus, $f(0) = g(0)$ and $f(1) = g(1)$, so for every $x \in \{0,1\}$, we have $f(x) = g(x)$. That is, $f = g$. QED 1-1

Onto: Given $(m,n) \in \mathbb{N} \times \mathbb{N}$, define $f : \{0,1\} \to \mathbb{N} \times \mathbb{N}$ by $f(0) = m \in \mathbb{N}$ and $f(1) = n \in \mathbb{N}$. Thus, f is indeed a function. Moreover, $G(f) = (f(0), f(1)) = (m,n)$. QED Onto

QED