Math 220-03, Fall 2025 Professor Rob Benedetto

Solutions to Homework #2

1. Section 1.1, Problem 2 (4 points): Which has the larger cardinality? The set of letters in the word *MISSISSIPPI* or the set of letters in the word *FLORIDA*?

Solution. The set for *FLORIDA* is larger. The set for *MISSISSIPPI* is $\{M, I, S, P\}$, which has cardinality 4, but The set for *FLORIDA* is $\{F, L, O, R, I, D, A\}$, which has cardinality $7 > 4$.

[Remember, of course, that we don't do repeats with elements of set; an object is either in the set or not, period.]

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2. Section 1.1, Problem 10(b(iv–ix)) (24 points): For each $k \in \{1, 2, \ldots, 20\}$, let $D_k = \{x \,|\, x \text{ is a prime dividing } k\}$. Let $\mathcal{D} = \{D_k | k = 1, 2, \ldots, 20\}$. True or false (and briefly justify):

(iv) $\varnothing \in \mathcal{D}$ (v) $\varnothing \subsetneq \mathcal{D}$ (vi) $5 \in \mathcal{D}$

(vii) $\{5\} \in \mathcal{D}$ (viii) $\{4, 5\} \in \mathcal{D}$ (ix) $\{\{3\}\} \subseteq \mathcal{D}$

Solution. (iv): True We have $D_1 = \varnothing$ (because no primes divide 1), so $\varnothing = D_1 \in \mathcal{D}$.

(v): True We have $\varnothing \subseteq \mathcal{D}$ because all elements of $\varnothing$ (all none of them) are in $\mathcal{D}$. But $\{5\} = D_5 \in \mathcal{D}$ but $\{5\} \notin \varnothing$. So $\varnothing \subsetneq \mathcal{D}$

(vi): False Every element of $\mathcal{D}$ is a *set* of numbers, but 5 is merely a number, not a set of numbers. So $5 \notin \mathcal{D}$.

(vii): True As noted in (v), we have $\{5\} = D_5 \in \mathcal{D}$.

(viii): False Every element of $\mathcal{D}$ is a set of *prime* numbers. Although $\{4, 5\}$ is a set of *numbers*, 4 is not prime, so $\{4, 5\} \notin \mathcal{D}$.

(ix): True We have $D_3 = \{3\}$, so $\{3\} \in \mathcal{D}$. Therefore, the set $\{\{3\}\}$, i.e., the set consisting of the one element $\{3\}$ [which happens to be a set], is *contained* in $\mathcal{D}$.

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3. Section 1.2, Problem 4(a,b,d,h,m,n) (18 points) Let U be the set of 52 cards in a standard deck. Let S, D, A, K be the sets of spades, diamonds, aces, and kings, respectively. Say which cards belong to each set below, and find the cardinality of each set. (And briefly justify.)

(a) $A \cap D$ (b) $S \cap D$ (d) $(A \cup K) \cap (S \cup D)$

(h) $K \cap [(S \cup D)^c]$ (m) $S \smallsetminus K$ (n) $K \smallsetminus S$

Solution. (a): $A \cap D$ consists of only the ace of diamonds. The cardinality is $|A \cap D| = 1$

(b): $S \cap D = \varnothing$ is the empty set, because there are no cards that are both spades and diamonds. The cardinality is $|S \cap D| = 0$

(d): $(A \cup K) \cap (S \cup D)$ consists of the ace and king of spades, and the ace and king of diamonds

The cardinality is $|(A \cup K) \cap (S \cup D)| = 4$ since there are two cards (ace and king) from each of the two suits (spades and diamonds).

(h): $K \cap [(S \cup D)^c]$ consists of the king of hearts and king of clubs

This is because $(S \cup D)^c$ consists of all cards that are *not* spades or diamonds, i.e., the clubs and hearts. Intersecting with the set of kings gives the set described above. The cardinality is $|K \cap [(S \cup D)^c]| = 2$

(m): $S \smallsetminus K$ consists of all spades *besides* the king which has cardinality $|S \smallsetminus K| = 12$

(n): $K \smallsetminus S$ consists of all kings that are not spades, i.e., $\boxed{\text{the kings of hearts, diamonds, and clubs}}$ which has cardinality $\boxed{|K \smallsetminus S| = 3}$

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4. Section 1.2, Problem 7(b), first two parts (6 points): Determine whether the sets in this collection are mutually disjoint. Also determine whether the collection is nested. (And of course, briefly justify everything!)

$$\mathcal{B} = \{(-\tfrac{1}{n}, n) \mid n \in \mathbb{N}\}$$

Solution. The sets are $\boxed{\text{not mutually disjoint}}$ because two of them intersect: $(-1,1) \cap (-\frac{1}{2}, 2) \neq \varnothing$ because it contains 0.
[Yes, it's true that *all* of the sets intersect, since they *all* contain 0, but to verify they are not mutually disjoint, it's required only to find two whose intersection is nonempty.]

No, the collection is $\boxed{\text{not nested}}$ because

$$(-1,1) \not\subseteq \Big(-\frac{1}{2}, 2\Big) \text{ since } -\frac{1}{2} \in (-1,1) \smallsetminus \Big(-\frac{1}{2}, 2\Big)$$

and

$$(-1,1) \not\supseteq \Big(-\frac{1}{2}, 2\Big) \text{ since } \frac{3}{2} \in \Big(-\frac{1}{2}, 2\Big) \smallsetminus (-1,1)$$

[Again, yes, it's true that *no* set in this collection contains any other, but to verify that they are not nested, it's required only to find two for which neither contains the other.]

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5. Section 1.2, Problem 7(b), last two parts (16 points): Find the union of the sets in the collection $\mathcal{B}$ above. Also find the intersection of the sets in $\mathcal{B}$. In both cases, say what the set is and also **prove** the equality of sets you are claiming.

Solution. Let $U = \bigcup_{S \in \mathcal{B}} S$ be the union, and $V = \bigcap_{S \in \mathcal{B}} S$ be the intersection.

Claim 1: $\boxed{\text{The union } U \text{ of the sets in } \mathcal{B} \text{ is } (-1,\infty)}$

Proof that $U = (-1,\infty)$:
($\subseteq$): Given $x \in U$, we have $x \in (-\frac{1}{n}, n)$ for some $n \in \mathbb{N}$. Then

$$-1 \leq -\frac{1}{n} < x < n < \infty,$$

so $x \in (-1,\infty)$.
($\supseteq$): Given $x \in (-1,\infty)$, we consider two cases.
If $x < 0$, then $x \in (-1,1) \subseteq U$. [This is because $(-1,1)$ is one of the sets $S \in \mathcal{B}$, for $n = 1$.]
Otherwise, we have $x \geq 0$. There is an integer $n \in \mathbb{N}$ such that $0 \leq x < n$, and hence $x \in (-\frac{1}{n}, n) \subseteq U$.
QED Claim 1

Claim 2: $\boxed{\text{The intersection } V \text{ of the sets in } \mathcal{B} \text{ is } [0,1)}$

Proof that $V = [0,1)$:
($\subseteq$): Given $x \in V$, we have $x \geq -\frac{1}{n}$ for every $n \in \mathbb{N}$. Therefore, $x \geq 0$.
In addition, $x \in V \subseteq (-1,1)$ because $(-1,1)$ is one of the sets in $\mathcal{B}$, so $x < 1$. Thus, $x \in [0,1)$.
($\supseteq$): Given $x \in [0,1)$, then for every $n \in \mathbb{N}$, we have

$$-\frac{1}{n} < 0 \leq x < 1 \leq n,$$

so $x \in (-\frac{1}{n}, n)$ for every $n \in \mathbb{N}$. Hence $x \in V$. QED Claim 2

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6. Section 1.2, Problem 8(d,e) (8 points): Determine whether the following statements are true or false, and briefly justify.

(d): $\mathcal{B} \subseteq \mathcal{A}$, where $\mathcal{B}$ is as in the previous problem, and $\mathcal{A} = \{(\frac{1}{n}, n+1) \mid n \in \mathbb{N}\}$

(e): $\mathcal{C} \subseteq \mathcal{D}$, where $\mathcal{C} = \{(n, \infty) \mid n \in \mathbb{N}\}$ and $\mathcal{D} = \{(x, \infty) \mid x \in \mathbb{R}\}$

Solution. (d): $\boxed{\text{False}}$ because $(-1, 1) \in \mathcal{B}$ (using $n = 1$) but $(-1, 1) \notin \mathcal{A}$ (because the left endpoint of each interval in $\mathcal{A}$ is positive, not negative).

[**Note**: In fact, *none* of the elements of $\mathcal{B}$ is an element of $\mathcal{A}$. But to prove $\mathcal{B} \not\subseteq \mathcal{A}$, we only need to come up with *one* element of $\mathcal{B}$ that is not in $\mathcal{A}$.]

(e): $\boxed{\text{True}}$

Any element U of $\mathcal{C}$ is of the form $U = (n, \infty)$ for some $n \in \mathbb{N}$. Therefore, since $n \in \mathbb{R}$, we have that $U \in \mathcal{D}$. So $\mathcal{C} \subseteq \mathcal{D}$, as claimed.