Solutions to Homework #19

1. Section 8.5, #4, variant: Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that the associated series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} a_n = 0$.

Proof, Method 1. For each $k \in \mathbb{N}$, let $s_k = a_1 + \cdots + a_k$. Let $S = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$.

Given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|s_n - S| < \varepsilon/2$.

Let $N = N_1 + 1$. Given $n \geq N$, we have $n, n-1 \geq N_1$, and hence $|s_{n-1} - S| < \varepsilon/2$ and $|s_n - S| < \varepsilon/2$. We also have $s_n = s_{n-1} + a_n$, and therefore

$$|a_n - 0| = |s_n - s_{n-1}| = |(s_n - S) + (S - s_{n-1})| \leq |s_n - S| + |s_{n-1} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{QED}$$

Proof, Method 2. For each $k \in \mathbb{N}$, let $s_k = a_1 + \cdots + a_k$. Let $S = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$.

We claim that $\lim_{n\to\infty} s_{n-1} = S$ as well. To see this, given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|s_n - S| < \varepsilon$.

Let $N = N_1 + 1 \in \mathbb{N}$. Given $n \geq N$, we have $n - 1 \geq N_1$, and hence $|s_{n-1} - S| < \varepsilon$, proving our claim.

Thus, by Theorem 8.3.9 [on the arithmetic of limits], we have

$\lim_{n\to\infty} a_n = a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0.$ QED

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2. Section 8.5, #6(b): Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be real sequences such that for all $n \in \mathbb{N}$, we have $0 \leq b_n \leq a_n$. If the series $\sum_{n=1}^{\infty} a_n$ converges, prove that the series $\sum_{n=1}^{\infty} b_n$ also converges.

Proof. For each $k \in \mathbb{N}$, let $s_k = \sum_{n=1}^{k} a_n$ and $t_k = \sum_{n=1}^{k} b_n$, the partial sums of the two series.

Let $L = \sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k$, so that $L \in \mathbb{R}$ since this series converges by hypothesis.

Note that the sequences $(s_k)_{k=1}^{\infty}$ and $(t_k)_{k=1}^{\infty}$ are both increasing, because for every $k \geq 1$, we have $s_{k+1} = s_k + a_{k+1} \geq s_k$, since $s_{k+1} \geq 0$, and similarly $t_{k+1} = t_k + b_{k+1} \geq t_k$, since $b_{k+1} \geq 0$.

Note also that for every $k \in \mathbb{N}$, we have $s_k \leq L$, since

$$L - s_k = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k} a_n = \sum_{n=k+1}^{\infty} a_n \geq \sum_{n=k+1}^{\infty} 0 = 0,$$

where the inequality is because $a_n \geq 0$ for every $n \in \mathbb{N}$.

It follows that for every $k \in \mathbb{N}$, we have

$$t_k = b_1 + b_2 + \cdots + b_k \leq a_1 + a_2 + \cdots + a_k = s_k \leq L.$$

Thus, $(t_k)_{k=1}^{\infty}$ is an increasing sequence that is bounded above. By the Monotone Sequence Theorem, the sequence $(t_k)_{k=1}^{\infty}$ converges. That is the series $\sum_{n=1}^{\infty} b_n$ converges. QED

Note: Here's an alternative proof that for every $k \in \mathbb{N}$, we have $s_k \leq L$:

Suppose not, i.e., that there is some m such that $s_m > L$. Let $\varepsilon = s_m - L > 0$. Then because $\lim_{k\to\infty} s_k = L$, there is some $N \in \mathbb{N}$ such that for all $k \geq N$, we have $|s_k - L| < \varepsilon$.

Let $k = \max(m, N)$. Then because $k \geq N$, we have

$$s_k = (s_k - L) + L \leq |s_k - L| + L < \varepsilon + L = (s_m - L) + L = s_m.$$

But because $k \geq m$ and the fact that $(s_k)_{k=1}^{\infty}$ is increasing, we have $s_k \geq s_m$, contradicting the above statement that $s_k < s_m$.
This contradiction proves our claim: for every $k \in \mathbb{N}$, we have $s_k \leq L$.

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3. Section 6.3, #3(a): Prove that $|(0,\infty)| = |\mathbb{R}|$.

Proof. Define $f : (0,\infty) \to \mathbb{R}$ by $f(x) = \ln x$. And define $g : \mathbb{R} \to (0,\infty)$ by $g(t) = e^t$.
Note that for all $x \in (0,\infty)$, from high school math we know that $\ln x \in \mathbb{R}$ is indeed defined. Similarly, for all $t \in \mathbb{R}$, we also know that $e^t \in (0,\infty)$. Thus, f and g are indeed functions.
For any $x \in (0,\infty)$, we have $g(f(x)) = e^{\ln x} = x$, and for any $t \in \mathbb{R}$, we have $f(g(x)) = \ln(e^t) = t$. Thus, f and g are inverses of one another. In particular, $f : (0,\infty) \to \mathbb{R}$ is bijective, so $|(0,\infty)| = |\mathbb{R}|$. QED

Note: There are lots of other ways to do this. There's no need to use the base-e logarithm. If you prefer, $f(x) = \log_2 x$ and $g(t) = 2^t$ will also work; or $\log_{10} x$ and 10^t; or in general $\log_a x$ and a^x for any constant $a > 1$.
One can also do this with a function like $f(x) = \dfrac{1}{x} - x$, which maps $(0,\infty)$ into $\mathbb{R}$ and which one can prove is bijective, with inverse $g(t) = \dfrac{-t + \sqrt{t^2+4}}{2}$.
There are many other ways, too.

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4. Section 6.3, #5(a): Let A, B be sets with $|A| = |B|$. Prove that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$

Proof. By hypothesis, there is a bijective function $f : A \to B$, which has an inverse function $g = f^{-1} : B \to A$.
Define $F : \mathcal{P}(A) \to \mathcal{P}(B)$ by $F(U) = f(U)$, and define $G : \mathcal{P}(B) \to \mathcal{P}(A)$ by $G(V) = g(V)$,

[That is, for any subset $U \subseteq A$, define $F(U)$ to be the subset of B that is the image $f(U)$ of U under f. Define G similarly.]

For any $U \in \mathcal{P}(A)$, we have that $U \subseteq A$, and hence $f(U) \subseteq B$, i.e., $F(U) = f(U) \in \mathcal{P}(B)$. So F is indeed a function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Similarly, G is indeed a function from $\mathcal{P}(B)$ to $\mathcal{P}(A)$.

For any $U \in \mathcal{P}(A)$, we have $G(F(U)) = g(f(U)) = g \circ f(U) = \mathrm{id}_A(U) = U$. That is, $G \circ F : \mathcal{P}(A) \to \mathcal{P}(A)$ is the identity function.

Similarly, for any $V \in \mathcal{P}(B)$, we have $F(G(V)) = f(g(V)) = f \circ g(V) = \mathrm{id}_B(V) = V$. That is, $F \circ G : \mathcal{P}(B) \to \mathcal{P}(B)$ is the identity function.

Hence, F is invertible (with inverse G) and hence bijective, so $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. QED

Note: Alternatively, one could define only F but not G and prove that F is bijective directly. Here's a proof along those lines, after defining F as above:

(**1-1**): Given $U_1, U_2 \in \mathcal{P}(A)$ with $F(U_1) = F(U_2)$, we claim that $U_1 = U_2$, which we now prove:

($\subseteq$): Given $x \in U_1$, we have $f(x) \in f(U_1) = F(U_1) = F(U_2) = f(U_2)$, so there is some $y \in U_2$ such that $f(x) = f(y)$. But f is 1-1, so $x = y \in U_2$. QED ($\subseteq$)

($\supseteq$): Similar, with the roles of U_1 and U_2 swapped. QED ($\supseteq$)
QED 1-1

(**onto**): Given $V \in \mathcal{P}(B)$, let $U = f^{-1}(V) \in \mathcal{P}(A)$. [That is, $U = \{x \in A \mid f(x) \in V\}$.] We claim that $F(U) = V$, as we now prove:

($\subseteq$): Given $y \in F(U) = f(U)$, there is some $x \in U$ such that $y = f(x)$. By definition of $U = f^{-1}(V)$, then, we have $y = f(x) \in V$. QED ($\subseteq$)

($\supseteq$): Given $y \in V$, then because f is onto, there is some $x \in A$ such that $f(x) = y$. Then $f(x) \in V$, so by definition of U, we have $x \in f^{-1}(V) = U$. So $y = f(x) \in f(U) = F(U)$. QED ($\supseteq$)
QED onto

Thus, F is bijective, so $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. QED