Solutions to Homework #18

1. Prove the following sort-of-converse to Theorem 8.4.2: Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be real sequences, let $L_1, L_2 \in \mathbb{R}$, and suppose that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$. Suppose further that $L_1 < L_2$. Prove that there is some $M \in \mathbb{N}$ such that for all $n \geq M$, we have $a_n < b_n$.

Proof. Let $\varepsilon = \dfrac{L_2 - L_1}{2} > 0$.
Since $\lim_{n\to\infty} a_n = L_1$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n - L_1| < \varepsilon$.
Since $\lim_{n\to\infty} b_n = L_s$, there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|b_n - L_2| < \varepsilon$.
Let $M = \max\{N_1, N_2\} \in \mathbb{N}$. Given any $n \geq M$, we have $|a_n - L_1| < \varepsilon$ and $|b_n - L_2| < \varepsilon$. In particular, $a_n - < \varepsilon$ and $L_2 - b_n < \varepsilon$. Therefore,

$$a_n - b_n = (a_n - L_1) + (L_1 - L_2) + (L_2 - b_n) < \varepsilon + (-2\varepsilon) + \varepsilon = 0,$$

and hence $a_n < b_n$. QED

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2. Show that the result of the previous problem fails badly if $L_1 = L_2$. More precisely, give an example of two real sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ and a real number $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, and so that for any $M \in \mathbb{N}$, there exist $m, n \geq M$ such that $a_m < b_m$ and $a_n > b_n$.

Proof. Define the two sequences by $a_n = \dfrac{-1^n}{n}$ and $b_n = \dfrac{-1^{n+1}}{n} = -a_n$.
We claim that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. To see this, given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $N > 1/\varepsilon$.
Given $n \geq N$, we have $|a_n - 0| = |a_n| = \dfrac{1}{n} < \varepsilon$ and $|b_n - 0| = |b_n| = \dfrac{1}{n} < \varepsilon$, proving our claim.
[Note: the claim can also be proven using the Squeeze Law.]
Given any $M \in \mathbb{N}$, pick integers $m, n \geq M$ with m odd and n even. Then $a_m < 0 < b_m$ and $a_n > 0 > b_n$. QED

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3. For every $k \in \mathbb{N}$, define $s_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$. Prove, by induction, that $s_k \leq 2 - \dfrac{1}{k}$ for every $k \in \mathbb{N}$.

Proof. **Base Case**: For $k = 1$, we have $s_1 = 1 = 2 - \dfrac{1}{1}$.
Ind. Step: Assume the statement is true for some $k = m$. Then

$$\begin{aligned} s_{m+1} = s_m + \frac{1}{(m+1)^2} &\leq 2 - \frac{1}{m} + \frac{1}{(m+1)^2} < 2 - \frac{1}{m} + \frac{1}{m(m+1)} \\ &= 2 - \left(\frac{(m+1) - 1}{m(m+1)}\right) = 2 - \frac{m}{m(m+1)} = 2 - \frac{1}{m+1}. \end{aligned}$$

QED

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4. Section 8.4, #3: Let $(a_n)_{n=1}^{\infty}$ be a real sequence that converges to $L \in \mathbb{R}$. Prove that the sequence $((-1)^n a_n)_{n=1}^{\infty}$ converges if and only if $L = 0$.

Proof. ($\Rightarrow$): Let $K = \lim_{n\to\infty}(-1)^n a_n$, which exists by assumption. The sequence $(a_{2n})_{n=1}^{\infty}$ is a subsequence of (a_n) and hence converges to the same limit L, by Theorem 8.4.5(a). But $(a_{2n})_{n=1}^{\infty} = ((-1)^{2n} a_{2n})_{n=1}^{\infty}$ is also a subsequence of $((-1)^n a_n)$ and hence converges to K, by the same theorem. Thus, $K = L$.
On the other hand, the sequence $(-a_{2n-1})_{n=1}^{\infty} = ((-1)^{2n-1} a_{2n-1})_{n=1}^{\infty}$ is a subsequence of $(-a_n)$ on the one hand, and of $((-1)^n a_n)$ on the other, so it converges both to $-L$ and to K. Thus, $L = K = -L$, so $L = 0$. QED ($\Rightarrow$)

($\Leftarrow$): We claim that $\lim_{n\to\infty}(-1)^n a_n = 0$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - L| < \varepsilon$.
Since $L = 0$, then given any $n \geq N$, we have $|(-1)^n a_n - 0| = |a_n| = |a_n - L| < \varepsilon$. QED ($\Leftarrow$)

Alternative Proof of ($\Rightarrow$): Let $K = \lim_{n\to\infty}(-1)^n a_n$, which exists by assumption. Suppose (towards a contradiction) that $L \neq 0$.

Let $\varepsilon = |L|/2 > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n - L| < \varepsilon$, and there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|(-1)^n a_n - K| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. Pick an even integer $n \geq N$. Then since $n, n+1 \geq N_1$ and $n, n+1 \geq N_2$, we have

$$|a_n - L|, |a_{n+1} - L|, |a_n - K|, |-a_{n+1} - K| < \varepsilon.$$

Therefore,

$$\begin{aligned}2|L| = |2L| &= |(L - a_n) + (a_n - K) + (K + a_{n+1}) + (L - a_{n+1})| \\ &\leq |a_n - L| + |a_n - K| + |-a_{n+1} - K| + |a_{n+1} - L| < \varepsilon + \varepsilon + \varepsilon + \varepsilon = 2|L|,\end{aligned}$$

which is a contradiction. Therefore, we must have $L = 0$. QED ($\Rightarrow$)

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5. Section 8.4, #6: Let $(a_n)_{n=1}^{\infty}$ be a decreasing real sequence. Show that $(a_n)_{n=1}^{\infty}$ either converges or else diverges to $-\infty$.

Proof. We consider two cases.

Case 1: The sequence has a lower bound, i.e., there is some $A \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $a_n \geq M$. Then since (a_n) is decreasing, by the Monotone Sequence Theorem, the sequence converges, as desired.

Case 2: Otherwise, the sequence does *not* have a lower bound. We claim in this case that the sequence diverges to $-\infty$.
To prove this claim, given any $M > 0$, then because $-M$ is not a lower bound, there is some $N \in \mathbb{N}$ such that $a_N < -M$. Given any $n \geq N$, we have $a_n \leq a_N$ because the sequence is decreasing. This proves the claim. QED