Math 220, Section 03, Fall 2025 Professor Rob Benedetto

Solutions to Homework #15

1. Section 8.1, #2
Give the first five terms, and the tenth term, of each of the following sequences:

(a) $(a_n)_{n=-3}^{\infty}$ where $a_n = (-2)^n$
(b) $(b_n)_{n=5}^{\infty}$ where $b_n = 12n - 121$
(c) $(c_n)_{n=-1}^{\infty}$ where $c_n = 2n^2 - n + 1$

Solutions. (a): The first five terms are when $n = -3, -2, -1, 0, 1$, and the tenth is five terms later, when $n = 6$.
So the first five terms are $\boxed{-\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, 1, -2}$ and the tenth is $\boxed{64}$

(b): The first five terms are when $n = 5, 6, 7, 8, 9$, and the tenth is five terms later, when $n = 14$.
So the first five terms are $\boxed{-61, -49, -37, -25, -13}$ and the tenth is $\boxed{47}$

(c): The first five terms are when $n = -1, 0, 1, 2, 3$, and the tenth is five terms later, when $n = 8$.
So the first five terms are $\boxed{4, 1, 2, 7, 16}$ and the tenth is $\boxed{121}$

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2. Section 8.1, #6 (a, c, d)
Let $(a_n)_{n=1}^{\infty}$ be the real sequence given by $a_n = 2n - 5$. For each function $f : \mathbb{N} \to \mathbb{N}$ below, find the sequence $(b_n)_{n=1}^{\infty}$ given by $b_n = a_{f(n)}$. Determine whether or not $(b_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. (And briefly explain why, of course.)

(a) $f(n) = n^2$
(c) $f(n) = \lfloor \frac{n}{2} \rfloor$
(d) $f(n) = |2n - 7|$

Solution/Proof. (a): We have $\boxed{b_n = 2n^2 - 5}$ for all $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$, we have

$$(n+1)^2 = n^2 + 2n + 1 > n^2, \qquad \text{so } f(n) = n^2 \text{ is strictly increasing.}$$

Thus, $\boxed{\textbf{yes}, (b_n)_{n=1}^{\infty} \text{ is a subsequence of } (a_n)_{n=1}^{\infty}}$

(c): We have $\boxed{b_n = 2\lfloor \frac{n}{2} \rfloor - 5}$ for all $n \in \mathbb{N}$. However, we have

$$f(3) = \left\lfloor \frac{3}{2} \right\rfloor = 1 = \left\lfloor \frac{2}{2} \right\rfloor = f(2),$$

so $f(n) = \lfloor \frac{n}{2} \rfloor$ is **not** strictly increasing. Thus, $\boxed{\textbf{no}, (b_n)_{n=1}^{\infty} \text{ is } \textbf{not} \text{ a subsequence of } (a_n)_{n=1}^{\infty}}$

Alternative reason for (c): Another reason why (b_n) is not a subsequence is that when $n = 1$, we get $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{1}{2} \rfloor = 0 \notin \mathbb{N}$.

(d): We have $\boxed{b_n = 2|2n - 7| - 5}$ for all $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$, we have

$$f(2) = |2(2) - 7)| = | - 3| = 3 < 5 = | - 5| = |2(1) - 7| = f(1),$$

so $f(n) = \lfloor \frac{n}{2} \rfloor$ is **not** strictly increasing. Thus, $\boxed{\textbf{no}, (b_n)_{n=1}^{\infty} \text{ is } \textbf{not} \text{ a subsequence of } (a_n)_{n=1}^{\infty}}$

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3. Section 8.1, variant of #7
Prove that a real sequence $(a_n)_{n=1}^{\infty}$ is both geometric and arithmetic if and only if it is constant.

Proof. ($\Rightarrow$) By hypothesis, there are constants $a, r, b, c \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, we have $a_n = ar^n$ and also $a_n = bn + c$.
We have $a_2 - a_1 = (2b + c) - (b + c) = b$ but also $a_2 - a_1 = ar^2 - ar = ar(r - 1)$. Thus, $ar(r - 1) = b$.
Similarly $a_3 - a_2 = (3b+c) - (2b+c) = b$ but also $a_3 - a_2 = ar^3 - ar^2 = ar^2(r-1)$. Thus, $ar^2(r-1) = b$.
Therefore, we have $ar^2(r - 1) = b = ar(r - 1)$, and hence $ar^2(r - 1) - ar(r - 1) = 0$, i.e., $ar(r - 1)^2 = 0$.
It follows that either $a = 0$ or $r = 0$ or $r = 1$.

Case 1: $a = 0$. Then for all $n \in \mathbb{N}$, we have $a_n = ar^n = 0$, so the sequence is constant (since all terms are 0).

Case 2: $r = 0$. Then for all $n \in \mathbb{N}$, we have $a_n = ar^n = 0$, so the sequence is constant (since all terms are 0).

Case 3: $r = 1$. Then for all $n \in \mathbb{N}$, we have $a_n = ar^n = a$, so the sequence is constant (since all terms are a).

($\Leftarrow$) Let $b = a_1$, so by assumption we have $a_n = c$ for all $n \in \mathbb{N}$. Define $r = 1 \in \mathbb{R}$ and $b = 0 \in \mathbb{R}$. Then for all $n \in \mathbb{N}$, we have

$a_n = c = cr^n$, so (a_n) is geometric, and $a_n = bn + c$, so (a_n) is arithmetic. QED

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4. Section 8.1, variant of #11
Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $(b_n)_{n=1}^{\infty}$ be a strictly increasing arithmetic sequence of positive integers, so that $(a_{b_n})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

(a) If $(a_n)_{n=1}^{\infty}$ is an arithmetic sequence, prove that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is also arithmetic.

(b) If $(a_n)_{n=1}^{\infty}$ is a geometric sequence, prove that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is also geometric.

Proof. (a): Since $\{a_n\}$ is arithmetic, there are constants $s, t \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $a_n = sn + t$. Thus, for all $n \in \mathbb{N}$, we have $a_{b_n} = a_{kn+m} = s(kn + m) + t = (sk)n + (sm + t)$. Since sk and $sm + t$ are real constants, it follows that $(a_{b_n})_{n=1}^{\infty}$ is an arithmetic sequence. QED (a)

(b): Since $\{a_n\}$ is geometric, there are constants $a, r \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $a_n = ar^n$. Thus, for all $n \in \mathbb{N}$, we have $a_{b_n} = a_{kn+m} = ar^{kn+m} = (ar^m) \cdot (r^k)^n$. Since ar^m and r^k are real constants, it follows that $(a_{b_n})_{n=1}^{\infty}$ is a geometric sequence. QED (b)

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5. Section 8.1, variant of #12
Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $(b_n)_{n=1}^{\infty}$ be a strictly increasing geometric sequence of positive integers, so that $(a_{b_n})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

If $(a_n)_{n=1}^{\infty}$ is a non-constant arithmetic sequence, prove that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is definitely **not** arithmetic.

Proof. Since (a_n) is arithmetic, there are constants $c, d \in \mathbb{R}$ such that $a_n = cn + d$ for all $n \in \mathbb{N}$. Since (b_n) is geometric, there are constants $b, r \in \mathbb{R}$ such that $b_n = br^n$ for all $n \in \mathbb{N}$.
Suppose, towards a contradiction, that (a_{b_n}) is arithmetic. Then $a_{b_3} - a_{b_2} = a_{b_2} - a_{b_1}$, so

$$cbr^2(r - 1) = cbr^3 - cbr^2 = (cb_3 + d) - (cb_2 + d) = (cb_2 + d) - (cb_1 + d) = cbr^2 - cbr = cbr(r - 1).$$

Rearranging, we have $cbr^2(r - 1) - cbr(r - 1) = 0$, i.e., $cbr(r - 1)^2 = 0$. Hence, one of these factors must be 0; we consider each case separately, as follows.

Case 1: $c = 0$. Then $a_n = d$ for all $n \in \mathbb{N}$, so that (a_n) is constant, a contradiction.

Case 2: $b = 0$. Then $b_n = 0$ for all $n \in \mathbb{N}$, so that (b_n) does not increase strictly, a contradiction.

Case 3: $r = 0$. Then $b_n = 0$ for all $n \in \mathbb{N}$, so that (b_n) does not increase strictly, a contradiction.

Case 4: $r - 1 = 0$, so that $r = 1$. Then $b_n = b$ for all $n \in \mathbb{N}$, so that (b_n) does not increase strictly, a contradiction.

Thus, in all cases, we have a contradiction. Therefore, our supposition must be false, and hence (a_{b_n}) is **not** arithmetic. QED