Solutions to Homework #14

1. Section 6.2, #4(a,b) For each of the following functions $f : A \to B$, decide whether or not it is invertible. If it is, find the inverse function.

(a): $A = B = \mathbb{R}$, and $f(x) = \dfrac{5x-1}{3}$

(b): $A = \mathbb{R}$, $B = [-4, \infty)$, and $f(x) = (x-3)^2 - 4$

Proof. (a): $\boxed{\text{Yes, } f \text{ is invertible}}$ and the inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$ by $f^{-1}(x) = \dfrac{3x+1}{5}$.
To prove this, first note that f^{-1} is indeed a function, since for any $x \in \mathbb{R}$, we have $(3x+1)/5 \in \mathbb{R}$. It remains to show the two compositions:
Given arbitrary $x \in A = \mathbb{R}$, we have

$$f^{-1}(f(x)) = f^{-1}\left(\frac{5x-1}{3}\right) = \frac{1}{5}\left[3\left(\frac{5x-1}{3}\right) + 1\right] = \frac{1}{5}\big[(5x-1)+1\big] = \frac{1}{5}(5x) = x.$$

Next, given arbitrary $x \in B = \mathbb{R}$, we have

$$f(f^{-1}(x)) = f\left(\frac{3x+1}{5}\right) = \frac{1}{3}\left[5\left(\frac{3x+1}{5}\right) - 1\right] = \frac{1}{3}\big[(3x+1)-1\big] = \frac{1}{3}(3x) = x.$$

QED (a)

(b): $\boxed{\text{No, } f \text{ is not invertible}}$ because it is not 1-1. To see this, note that $4, 2 \in A = \mathbb{R}$ with $4 \neq 2$, but

$$f(4) = (4-3)^2 - 4 = -3 = (2-3)^2 - 4 = f(2).$$

QED (b)

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2. Section 6.2, #4(c), variant

Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(n) = \begin{cases} 2n & \text{if } n \geq 1, \\ -2n+1 & \text{if } n \leq 0. \end{cases}$

Prove that f is invertible, with inverse function $f^{-1} : \mathbb{N} \to \mathbb{Z}$ given by

$$f^{-1}(m) = \begin{cases} m/2 & \text{if } m \text{ is even,} \\ -(m-1)/2 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. First, given arbitrary $n \in \mathbb{Z}$, we must show $f^{-1}(f(n)) = n$. We consider two cases:

Case 1: $n \geq 1$. Then

$$f^{-1}(f(n)) = f^{-1}(2n) = \frac{2n}{2} = n,$$

since $2n$ is an even integer.

Case 2: $n \not\geq 1$. Then $n \leq 0$, so

$$f^{-1}(f(n)) = f^{-1}(-2n+1) = -\frac{((-2n+1)-1)}{2} = -\left(\frac{-2n}{2}\right) = n,$$

since $-2n+1$ is an odd integer.

Second, given arbitrary $m \in \mathbb{N}$, we must show $f(f^{-1}(m)) = m$. We again consider two cases:

Case 1: m is even. Then

$$f(f^{-1}(m)) = f\left(\frac{m}{2}\right) = 2 \cdot \frac{m}{2} = m,$$

since $m \geq 2$, so $m/2 \geq 1$.

Case 2: m is odd. Then

$$f(f^{-1}(m)) = f\left(-\frac{(m-1)}{2}\right) = -2\left(-\frac{(m-1)}{2}\right) + 1 = (m-1) + 1 = m,$$

since $m \geq 1$, so $m - 1 \geq 0$, and hence $-(m-1)/2 \leq 0$. QED

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3. Section 6.2, #6(a,b)
Let $g : \mathbb{R} \to \mathbb{Z}$ be the *ceiling function* $g(x) = \lceil x \rceil$, defined to be the smallest integer that is greater than or equal to x.

(a) Prove that the image $g([-\pi, \pi])$ is $g([-\pi, \pi]) = \{-3, -2, -1, 0, 1, 2, 3, 4\}$.

(b) Prove that the inverse image $g^{-1}(\{-2, -1\})$ is $g^{-1}(\{-2, -1\}) = (-3, -1]$.

Proof. (a): ($\subseteq$): Given $y \in g([-\pi, \pi])$, there is some $x \in [-\pi, \pi]$ such that $y = g(x) = \lceil x \rceil$. Since $x \geq -\pi$, we have $y = \lceil x \rceil \geq \lceil -\pi \rceil = -3$. Similarly, since $x \leq \pi$, we have $y = \lceil x \rceil \leq \lceil \pi \rceil = 4$. Thus, $y \in \mathbb{Z}$ with $-3 \leq y \leq 4$, so $y \in$ RHS QED ($\subseteq$)

($\supseteq$): Given $y \in$ RHS, we consider two cases:

Case 1: $y = 4$. Then $y = \lceil \pi \rceil = g(\pi) \in g([-\pi, \pi])$, as desired.

Case 2: $y \neq 4$. Then $y \in \{-3, -2, -1, 0, 1, 2, 3\} = [-\pi, \pi] \cap \mathbb{Z}$. Thus, $y = \lceil y \rceil = g(y) \in g([-\pi, \pi])$. QED

(b): ($\subseteq$): Given $x \in g^{-1}(\{-2, -1\})$, we have $x \in \mathbb{R}$ and $g(x) \in \{-1, -2\}$, so we have two cases:

Case 1: If $g(x) = -1$, then $x \leq \lceil x \rceil = -1$. We also have $x + 1 > \lceil x \rceil = -1$, and hence $x > -2 > -3$. Thus, $x \in (-3, -1]$.

Case 2: If $g(x) = -2$, then $x \leq \lceil x \rceil = -2 \leq -1$. We also have $x + 1 > \lceil x \rceil = -2$, and hence $x > -3$. Thus, $x \in (-3, -1]$. QED ($\subseteq$)

($\supseteq$): Given $x \in (-3, -1]$, we have $x \in \mathbb{R}$ with $-3 < x \leq -1$. We consider two cases:

Case 1: If $x \leq -2$, then $-3 < x \leq -2$, and hence -2 is an integer $\geq x$, and it is the smallest such, since $-3 \not\geq x$. Thus, $g(x) = -2 \in \{-1, -2\}$, and hence $x \in g^{-1}(\{-2, -1\})$.

Case 2: Otherwise, $-2 < x \leq -1$, and hence -1 is an integer $\geq x$, and it is the smallest such, since $-2 \not\geq x$. Thus, $g(x) = -1 \in \{-1, -2\}$, and hence $x \in g^{-1}(\{-2, -1\})$. QED

4.Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x^2 - 7$. Prove that:

(a) $f^{-1}\big([2, 5)\big) = \left(-\sqrt{6}, -\frac{3}{\sqrt{2}}\right] \cup \left[\frac{3}{\sqrt{2}}, \sqrt{6}\right)$

(b) $f^{-1}\big((-10, -5]\big) = [-1, 1]$

(c) $f\big((-3, 2)\big) = [-7, 11)$

Proof. (a): ($\subseteq$): Given $x \in f^{-1}\big([2, 5)\big)$, we have $x \in \mathbb{R}$ with $f(x) \in [2, 5)$. That is, $2 \leq 2x^2 - 7 < 5$, so $9 \leq 2x^2 < 12$, and hence $9/2 \leq x^2 < 6$. Taking square roots gives $3/\sqrt{2} \leq |x| < \sqrt{6}$. Thus, $x \in$ RHS. QED ($\subseteq$)

($\supseteq$): Given $x \in$ RHS, we have $3/\sqrt{2} \leq |x| < \sqrt{6}$, so $9/2 \leq x^2 < 6$. Hence, $9 \leq 2x^2 < 12$, and so $2 \leq 2x^2 - 7 < 5$. Thus, $x \in \mathbb{R}$ with $f(x) \in [2, 5)$, i.e., $x \in$ LHS. QED (a)

(b): ($\subseteq$): Given $x \in$ LHS, we have $x \in \mathbb{R}$ with $f(x) \in (-10, -5]$. In particular $2x^2 - 7 \leq -5$, so $2x^2 \leq 2$, and hence $x^2 \leq 1$. Thus, $-1 \leq x \leq 1$. That is, $x \in [-1, 1]$. QED ($\subseteq$)
($\supseteq$): Given $x \in [-1, 1]$, we have $|x| \leq 1$ and hence $x^2 \in [0, 1]$. Thus, $2x^2 \in [0, 2]$, and therefore $f(x) = 2x^2 - 7 \in [-7, -5] \subseteq (-10, -5]$. That is, $x \in$ LHS. QED (b)

(c): ($\subseteq$): Given $y \in$ LHS, there is some $x \in (-3, 2)$ such that $f(x) = y$. Thus, $y = 2x^2 - 7 \geq 0 - 7 = -7$. On the other hand, since $|x| < 3$, we have $x^2 < 9$, and hence $y = 2x^2 - 7 < 18 - 7 = 11$. hence, $y \in [-7, 11)$.
($\supseteq$): Give $y \in [-7, 11)$, let $x = -\sqrt{(y + 7)/2}$. Note that x is defined, because $y \geq 7$, and hence $(y + 7)/2 \geq 0$. In addition, since the square root function outputs nonnegative numbers, we have $x \leq 0$. Also, $(y + 7)/2 < (11 + 7)/2 = 9$, so that $\sqrt{(y + 7)/2} < \sqrt{9} = 3$, and hence $x > -3$. Thus, $x \in (-3, 0] \subseteq (-3, 2)$. Therefore $y \in$ LHS. QED