Solutions to Homework #13

1. (18 points) Let $X = \mathbb{R} \setminus \{5\}$ and $Y = \mathbb{R} \setminus \{2\}$. Define $f : X \to Y$ by $f(x) = \dfrac{2x-3}{x-5}$.

   (a) (5 points): Prove that f is defined, i.e., for every $x \in X$, the value $f(x)$ is defined and is in Y.

   (b) (5 points): Prove that f is one-to-one.

   (c) (8 points): Prove that f is onto.

Proof. (a): Given $x \in X$, then because $x \neq 5$, we have $x - 5 \neq 0$, and hence $f(x) = (2x-3)/(x-5) \in \mathbb{R}$. It remains to show that $f(x) \in Y$, i.e., that $f(x) \neq 2$.
Suppose, toward contradiction, that $f(x) = 2$. Then $(2x-3)/(x-5) = 2$, so $2x - 3 = 2(x-5)$, i.e., $2x - 3 = 2x - 10$, and hence $7 = 0$, a contradiction. Thus, $f(x) \neq 2$, and hence $f(x) \in Y$. QED (a)

(b): Given $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$, we have $\dfrac{2x_1-3}{x_1-5} = \dfrac{2x_2-3}{x_2-5}$, and hence $(2x_1-3)(x_2-5) = (2x_2-3)(x_1-5)$. That is, $2x_1x_2 - 3x_2 - 10x_1 + 15 = 2x_1x_2 - 3x_1 - 10x_2 + 15$. Rearranging gives $7x_2 = 7x_1$, and hence $x_2 = x_1$. QED

(c): Given $y \in Y$, let $x = \dfrac{5y-3}{y-2}$, which is defined and in $\mathbb{R}$ because $y \neq 2$. We claim that $x \neq 5$.
To see this, suppose not; then $\dfrac{5y-3}{y-2} = 5$, so $5y - 3 = 5(y-2)$. Thus, $5y - 3 = 5y - 10$, so that $7 = 0$, a contradiction, proving our claim. That is, $x \in \mathbb{R} \setminus \{5\} = X$. Finally, we compute

$$f(x) = f\left(\frac{5y-3}{y-2}\right) = \frac{2\left(\dfrac{5y-3}{y-2}\right) - 3}{\left(\dfrac{5y-3}{y-2}\right) - 5} = \frac{2(5y-3) - 3(y-2)}{(5y-3) - 5(y-2)} = \frac{10y - 6 - 3y + 6}{5y - 3 - 5y + 10} = \frac{7y}{7} = y \quad \text{QED}$$

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2. (5 points)
Let $f : A \to B$ be a function, and let $C \subseteq A$ be a subset. Define $g : C \to A$ by $g(x) = x$.
Prove that $f|_C = f \circ g$

Proof. Both $f \circ g$ and $f|_C$ are functions with domain C and target set B. Given any $x \in C$, we have

$$f \circ g(x) = f\big(g(x)\big) = f(x) = f|_C(x). \quad \text{QED}$$

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3. (10 points) Let $A \neq \varnothing$ be a nonempty set. Define $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$.

   (a) (5 points): Prove that f is one-to-one.

   (b) (5 points): Prove that f is **not** onto.

Proof. (a): Given $x_1, x_2 \in A$ with $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Thus, $x_1 \in \{x_1\} = \{x_2\}$, so $x_1 = x_2$. QED (a)

(b): Let $S = \varnothing \in \mathcal{P}(A)$. It suffices to show that for all $x \in A$, we have $f(x) \neq S$, as follows:
Given $x \in A$, we have $f(x) = \{x\} \neq \varnothing = S$. QED (b)

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4. (16 points)
Let B be a set, and define $g : \mathcal{P}(B) \to \mathcal{P}(B)$ by $g(S) = B \setminus S$.

   (a) (4 points): Prove that g is indeed a function.

   (b) (5 points): Prove that for every $S \in \mathcal{P}(B)$, we have $g \circ g(S) = S$.

   (c) (7 points): Prove that g is bijective.

Proof. (a): Given $S \in \mathcal{P}(B)$, we have that $B \setminus S \subseteq B$, and hence $g(S) = B \setminus S \in \mathcal{P}(B)$. QED (a)

(b): Given $S \in \mathcal{P}(B)$, we claim that $B \setminus (B \setminus S) = S$.

Proof of Claim: ($\subseteq$): Given $x \in B \smallsetminus (B \smallsetminus S)$, we have $x \in B$ and $x \notin B \smallsetminus S$. That is, x is an element of B, but it is false that x is not in S; that is, $x \in S$. QED ($\subseteq$)

($\supseteq$): Given $x \in S$, we have $x \in B$, and it is false that $x \notin S$. That is, we have $x \in B$ and $x \notin B \smallsetminus S$. Hence, $x \in B \smallsetminus (B \smallsetminus S)$. QED ($\supseteq$)

QED Claim

Hence, by the claim, we have

$$g \circ g(S) = g(g(S)) = g(B \smallsetminus S) = B \smallsetminus (B \smallsetminus S) = S.$$ QED (b)

(c): **(1-1)**: Given $S_1, S_2 \in \mathcal{P}(B)$ such that $g(S_1) = g(S_2)$, we have

$$S_1 = g(g(S_1)) = g(g(S_2)) = S_2,$$

where the first and third equalities are by (b), and the second is by assumption.

(Onto): Given $T \in \mathcal{P}(B)$, let $S = g(T) \in \mathcal{P}(B)$. Then by (b), we have

$$g(S) = g(g(T)) = g \circ g(T) = T.$$ QED (c)

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5. Section 6.2, Problem 14 (12 points)
Prove Theorem 6.2.8(b): Let $f : A \to B$ be a function, and let $C, D \subseteq A$ be subsets. Prove that $f(C \cup D) = f(C) \cup f(D)$

Proof. ($\subseteq$): Given $y \in f(C \cup D)$, there is some $x \in C \cup D$ such that $f(x) = y$.
Case 1: If $x \in C$, then $y \in f(C) \subseteq f(C) \cup f(D)$.
Case 2: If $x \in D$, then $y \in f(D) \subseteq f(C) \cup f(D)$.

($\supseteq$): Given $y \in f(C) \cup f(D)$.
Case 1: If $y \in f(C)$, there is some $x \in C$ such that $f(x) = y$. Since $x \in C \cup D$, we have $y \in f(C \cup D)$.
Case 2: If $y \in f(D)$, there is some $x \in D$ such that $f(x) = y$. Since $x \in C \cup D$, we have $y \in f(C \cup D)$.
QED

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6. (15 points) Define $h : [-4, 4] \to [3, 7]$ by $h(x) = 3 + \sqrt{16 - x^2}$. Prove that
   (a) (5 points): h is indeed a function.
   (b) (6 points): h is onto.
   (c) (4 points): h is **not** one-to-one.

Proof. (a): Given $x \in [-4, 4]$, we have $0 \leq x^2 \leq 16$, and hence $0 \leq 16 - x^2 \leq 16$. Thus, $\sqrt{16 - x^2}$ is defined and lies between 0 and 4. Hence, $h(x)$ is defined, with $3 \leq h(x) \leq 7$. QED (a)

(b): Given $y \in [3, 7]$, let $x = \sqrt{16 - (y-3)^2}$. To see that $x \in [-4, 4]$, note that $y - 3 \in [0, 4]$, and hence $16 - (y-3)^2 \in [0, 16]$. Thus, x is defined, with $0 \leq x \leq 4$; so in particular, $x \in [-4, 4]$. Moreover,

$$h(x) = 3 + \sqrt{16 - [16 - (y-3)^2]} = 3 + \sqrt{(y-3)^2} = 3 + |y - 3| = y,$$

where the last equality is because $y \geq 3$, and hence $|y - 3| = y - 3$. QED (b)

(c): We have $\pm 4 \in [-4, 4]$, with $h(-4) = 3 = h(4)$, but $-4 \neq 4$. QED (c)