Solutions to Homework #12

1. Section 6.1, Problem 11(a,b) (20 points)
Define $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ by $f(n) = \{n, 2n, 3n, \ldots\} = \{kn \,|\, k \in \mathbb{N}\}$

Define $g : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ by $g(A) = \begin{cases} 1 & \text{if } A = \varnothing, \\ \min(A) & \text{if } A \neq \varnothing \end{cases}$

(a): Is f injective? Is f surjective? Prove your answers.

(b): Is g injective? Is g surjective? Prove you answers.

Solutions/Proofs. (a) $\boxed{f \text{ is injective but not surjective}}$

Injective: Given $m, n \in \mathbb{N}$ with $f(m) = f(n)$, we have $\{m, 2m, 3m, \ldots\} = \{n, 2n, 3n, \ldots\}$. Taking the minimum element of each of these two sets [which are the same set, by assumption], we have $m = n$. QED 1-1

Not surjective: Let $A = \varnothing \in \mathcal{P}(\mathbb{N})$. For any $n \in \mathbb{N}$, we have $n \in \{n, 2n, 3n, \ldots\} = f(n)$, and hence $f(n) \neq \varnothing = A$. QED not onto

[**Note**: There are *many* sets $A \in \mathcal{P}(\mathbb{N})$ that could be chosen to prove that f is not surjective.]

(b): $\boxed{g \text{ is surjective but not injective}}$

Surjective: Given $n \in \mathbb{N}$, let $A = \{n\} \in \mathcal{P}(\mathbb{N})$. Then $g(A) = n$ QED onto

Not injective: Let $A = \{1\} \in \mathcal{P}(\mathbb{N})$ and $B = \{1, 2\} \in \mathcal{P}(\mathbb{N})$. Then $A \neq B$, but $g(A) = 1 = g(B)$. QED not 1-1

[**Note**: For the surjectivity proof, for any $n \in \mathbb{N}$, there are many ways to choose $A \in \mathcal{P}(\mathbb{N})$ And for the non-injectivity, there are also many ways to choose A and B.]

---

2. Section 6.1, Problem 11(c,d), variant (12 points)
For f and g as in the previous problem:

(c): Prove that for every $n \in \mathbb{N}$, we have $g \circ f(n) = n$.

(d): Give three different examples of sets $A \in \mathcal{P}(\mathbb{N})$ such that $f \circ g(A) \neq A$.

Proofs. (c): Given $n \in \mathbb{N}$, we have

$$f \circ g(n) = f\big(\{n, 2n, 3n, \ldots\}\big) = n \qquad \text{QED}$$

(d) Let $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$. Then each $A_i \in \mathcal{P}(\mathbb{N})$, and

$$g \circ f(A_i) = g(i) = \{i, 2i, 3i, \ldots\} \neq A_i \qquad \text{QED}$$

[**Note**: Once again, there are just so many ways to choose sets A here that work.]

---

3. Section 6.1, Problem 13 (10 points)
Prove that composition of functions is associative. That is:
for any functions $f : A \to B$, $g : B \to C$, and $h : C \to D$, prove that $h \circ (g \circ f) = (h \circ g) \circ f$

Proof. First observe that $g \circ f$ is a function from A to C, so $h \circ (g \circ f)$ is a function from A to D. Similarly, $h \circ g$ is a function from B to D, so $(h \circ g) \circ f$ is also a function from A to D. It remains to show that the two functions do the same thing to each element of their common domain.
Given $x \in A$,

$$h \circ (g \circ f)(x) = h\big(g \circ f(x)\big) = h\Big(g\big(f(x)\big)\Big) = h \circ g\big(f(x)\big) = (h \circ g) \circ f(x).$$

Since this holds for all $x \in A$, we have $h \circ (g \circ f) = (h \circ g) \circ f$. QED

---

4. Section 6.1, Problem 16 (12 points)
Prove Theorem 6.1.14(b): for any functions $f : A \to B$ and $g : B \to C$, if f and g are both onto, then $g \circ f : A \to C$ is also onto.

Proof. Given $c \in C$, there exists $b \in B$ such that $g(b) = c$, since g is onto. So there exists $a \in A$ such that $f(a) = b$, since f is onto. So $g \circ f(a) = g(f(a)) = g(b) = c$. QED

---

5. Section 6.1, Problem 17(a) (15 points)
Let $f : A \to B$ and $g : B \to C$ be functions.

(i): If $g \circ f$ is onto, prove that g is onto.

(ii): Show the converse of (i) fails. That is, give examples of functions f, g as above for which g is onto, but $g \circ f$ is not onto. (And prove your claims, of course.)

Solutions/Proofs. (i): Given $c \in C$, there exists $a \in A$ such that $g \circ f(a) = c$, since $g \circ f$ is onto. Let $b = f(a) \in B$. Then $g(b) = g(f(a)) = g \circ f(a) = c$ QED (i)

(ii): Let $A = B = C = \mathbb{R}$, and define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ and $g(x) = x$. Then g is onto, because for all $y \in \mathbb{R}$, we have $g(y) = y$. However, $g \circ f(x) = g(0) = 0$ for all $x \in \mathbb{R}$. Thus, $g \circ f$ is not onto, because (for example) $5 \in \mathbb{R}$ is not in the image; for every $x \in \mathbb{R}$, we have $g \circ f(x) = 0 \neq 5$. QED (ii)

[**Note**: there are *many* ways to do this. Another example would be to pick $A = \{1\}$ and $B = C = \{1, 2\}$, with $f : A \to B$ by $f(1) = 1$, and $g : B \to C$ by $g(x) = x$.]

---

6. Section 6.1, Problem 17(b) (15 points)
Let $f : A \to B$ and $g : B \to C$ be functions.

(i): If $g \circ f$ is one-to-one, prove that f is one-to-one.

(ii): Show the converse of (i) fails. That is, give examples of functions f, g as above for which f is one-to-one, but $g \circ f$ is not one-to-one. (And prove your claims, of course.)

Solutions/Proofs. (i): Given $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, then we have $g(f(a_1)) = g(f(a_2))$, and hence $g \circ f(a_1) = g \circ f(a_2)$. Therefore, since $g \circ f$ is 1-1, we have $a_1 = a_2$. QED (i)

(ii): Let $A = B = C = \mathbb{R}$, and define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ and $g(x) = 0$. Then f is 1-1, because for all $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$, we have $x_1 = x_2$. However, $g \circ f(x) = g(x) = 0$ for all $x \in \mathbb{R}$. Thus, $g \circ f$ is not 1-1, because (for example) $5 \neq 4 \in \mathbb{R}$ but $g \circ f(5) = 0 = g \circ f(4)$. QED (ii)

[**Note**: again, there are *many* ways to do this. Another example would be to pick $A = B = \{1, 2\}$ and $C = \{1\}$, with $f : A \to B$ by $f(x) = x$, and $g : B \to C$ by $g(x) = 1$.]