Solutions to Homework #11

1. Section 6.1, Problem 3 (20 points)
Let $S = \{1, 2, 3, 4\}$. For each of the following subsets of $S \times S$, determine whether it is a function. For each one that is a function, find its range, determine whether it is one-to-one, and determine whether it is onto.

(a): $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$ (b): $\{(1, 1), (3, 1), (2, 1), (4, 1)\}$

(c): $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$ (d): $\{(1, 3), (2, 4), (3, 3), (4, 3)\}$ (e): $\{(1, 3), (2, 1), (3, 2)\}$

Solutions. (a): $\boxed{\text{Yes, function}}$ because each element of $S = \{1, 2, 3, 4\}$ shows up as the first coordinate of exactly one ordered pair in the set.
$\boxed{\text{Range is } \{1, 2, 3, 4\}}$ because all four of these elements show up as second coordinates. Thus, $\boxed{\text{yes, onto}}$.
Finally, $\boxed{\text{yes, one-to-one}}$ because no two pairs have the same second coordinate. [Alternatively, this is because the function is onto, and domain and target are both the same finite cardinality.]

(b): $\boxed{\text{Yes, function}}$ because each element of $S = \{1, 2, 3, 4\}$ shows up as the first coordinate of exactly one ordered pair in the set.
$\boxed{\text{Range is } \{1\}}$ because 1 is the only element that appears as a second coordinate. Since (for example) $2 \in S$ does not show up, $\boxed{\text{no, not onto}}$.
Finally, $\boxed{\text{no, not one-to-one}}$ because the pairs $(1, 1)$ and $(3, 1)$ both show up with the same second coordinate but different first coordinates. [Alternatively, this is because the function is not onto, and domain and target are both the same finite cardinality.]

(c): $\boxed{\text{No, not function}}$ because $(1, 1)$ and $(1, 2)$ have the same first coordinate but different second coordinates.

(d): $\boxed{\text{Yes, function}}$ because each element of $\{1, 2, 3, 4\}$ shows up as the first coordinate of exactly one ordered pair in the set.
$\boxed{\text{Range is } \{3, 4\}}$ because these are the only two elements that appear as second coordinates. Since (for example) $1 \in S$ does not show up, $\boxed{\text{no, not onto}}$.
Finally, $\boxed{\text{no, not one-to-one}}$ because the pairs $(1, 3)$ and $(3, 3)$ both show up with the same second coordinate but different first coordinates. [Alternatively, this is because the function is not onto, and domain and target are both the same finite cardinality.]

(e): $\boxed{\text{No, not function}}$ because $4 \in S$ is not the first coordinate of any pair.

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2. Section 6.1, Problem 4(a) (15 points)
Let $f : [0, \infty) \to (0, 1]$ by $f(x) = \dfrac{1}{x + 1}$. You may assume that f is a function. Determine whether it is one-to-one, onto, neither, or both. If it is not onto, determine its range.

Solution/Proof. $\boxed{f \text{ is } \textbf{both} \text{ one-to-one and onto}}$
1-1: Given $x_1, x_2 \in [0, \infty)$ with $f(x_1) = f(x_2)$, we have

$$\frac{1}{x_1 + 1} = \frac{1}{x_2 + 1}, \qquad \text{so} \qquad x_1 + 1 = x_2 + 1, \qquad \text{so} \qquad x_1 = x_2. \qquad \text{QED 1-1}$$

Onto: Given $y \in (0, 1]$, let $x = \dfrac{1}{y} - 1$, which is defined and in $\mathbb{R}$ because $y \neq 0$. In addition, we have

$$\frac{1}{y} > 0 \quad \text{since} \quad y > 0, \qquad \text{and so} \qquad \frac{1}{y} \geq 1 \quad \text{since} \quad y \leq 1.$$

Thus, $\dfrac{1}{y} \in [1, \infty)$, and hence $x = \dfrac{1}{y} - 1 \in [0, \infty)$.

Finally, $f(x) = \dfrac{1}{(\frac{1}{y} - 1) + 1} = \dfrac{1}{1/y} = y$. QED

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3. Section 6.1, Problem 4(b) (12 points)
Let $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \sin x$. You may assume that s is a function. Determine whether it is one-to-one, onto, neither, or both. If it is not onto, determine its range.

Solution/Proof. $\boxed{s \text{ is } \textbf{neither} \text{ one-to-one nor onto}}$

Not 1-1: We have $0, \pi \in \mathbb{R}$ and $s(0) = 0 = s(\pi)$ but $0 \neq \pi$. QED not 1-1
[There are, of course, *many* examples of x-values $a \neq b \in \mathbb{R}$ such that $\sin(a) = \sin(b)$; this is merely one such choice.]

Not onto: Let $y = 2 \in \mathbb{R}$. For all $x \in \mathbb{R}$, we have $-1 \leq s(x) \leq 1$, so $s(x) \neq y$ QED not onto
We know from basic trigonometry that $\sin(x)$ takes on every value in $[-1, 1]$, so $\boxed{\text{the range of } s \text{ is } [-1, 1].}$

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4. Section 6.1, Problem 4(c) (18 points)
Let $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = \lfloor (n^2 + 3)/n \rfloor$. You may assume that g is a function. Determine whether it is one-to-one, onto, neither, or both. If it is not onto, determine its range.

Solution/Proof. $\boxed{g \text{ is } \textbf{neither} \text{ one-to-one nor onto}}$
Note that $g(n) = \lfloor n + (3/n) \rfloor$.

Not 1-1: We have $1, 3 \in \mathbb{N}$ with $1 \neq 3$ but $g(1) = \lfloor 1 + 3 \rfloor = 4$ and $g(3) = \lfloor 3 + 1 \rfloor = 4$. QED not 1-1
[FYI: It turns out $g(4) = 4$ also. But besides $g(1) = g(3) = g(4)$, there are no other distinct choices $a \neq b \in \mathbb{N}$ such that $g(a) = g(b)$.]

Not onto: Let $y = 1 \in \mathbb{N}$. Given any $x \in \mathbb{N}$, we claim that $g(x) \neq y$. To see this, note that if $x = 1$, we already saw that $g(1) = 4 \neq 1$. Otherwise, we have $x \geq 2$, in which case $g(x) \geq \lfloor x \rfloor = x \geq 2$, so $g(x) \neq 1$. QED not onto

We also have that $g(2) = \lfloor 2 + 1.5 \rfloor = 3$, and as we saw above, $g(3) = 4$. We claim that for all $x \in \mathbb{N}$ with $x \geq 4$, we have $g(x) = x$. To see this, given such x, we have $0 < 3/x < 1$, and hence $g(x) = \lfloor x + (3/x) \rfloor = x$, as desired.
Thus, g takes on every value in $\{y \in \mathbb{N} \mid y \geq 4\}$, since we have $g(y) = y$ for all such y. We also saw $g(2) = 3$, so in fact, g takes on every value in $\{y \in \mathbb{N} \mid y \geq 3\}$. Recalling that $g(1) = 4$, it follows that $\boxed{\text{the range of } g \text{ is } \{y \in \mathbb{N} \mid y \geq 3\}}$

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5. Section 6.1, Problem 5(a) (15 points)
Consider the function $g : \{n \in \mathbb{N} \mid n \geq 100\} \to \mathbb{N}$ by $g(n) =$ the sum of the digits of n. Determine whether g is one-to-one, onto, neither, or both. If it is not onto, determine its range.

Solution/Proof. $\boxed{g \text{ is } \textbf{onto} \text{ but } \textbf{not one-to-one}}$

Onto: Given $b \in \mathbb{N}$, let n be the integer given by b 1's followed by two 0's, i.e., $n = \underbrace{11 \cdots 1}_{b}00$. Then since $b \geq 1$, we have that n is an integer with $n \geq 100$, so n is in the domain of g. By definition of g, we have $g(n) = b$. QED Onto

Not 1-1: We have $200, 1100 \in \{n \in \mathbb{N} \mid n \geq 100\}$ with $200 \neq 1100$, but $g(200) = 2 = g(1100)$. QED not 1-1
[There are **lots** of other ways to see that g is not one-to-one, of course.]