Solutions to Homework #10

1. (16 points) Prove that $\bigcap_{t\in(2,6)} [0, 2t + 5) = [0, 9]$

Proof. ($\subseteq$): Given $x \in$LHS, then using $t = 3 \in (2, 6)$, we have $x \in [0, 2 \cdot 3 + 5) = [0, 11)$, so that in particular, $x \in \mathbb{R}$ with $0 \le x < 11$. It remains to show that $x \le 9$.
Suppose (towards a contradiction) that $x > 9$. Let $t = (x - 5)/2$. Then since $9 < x < 11$, we have $4 < x - 5 < 6$, so

$$2 = \frac{4}{2} < t = \frac{x-5}{2} < \frac{6}{2} < 6,$$

and hence $t \in (2, 6)$. But we also have $2t+5 = (x-5)+5 = x$, so $x = 2t+5 \notin [0, 2t+5)$, a contradiction. Thus, we must have $x \le 9$.
Since $x \in \mathbb{R}$ with $0 \le x \le 9$, we have $x \in [0, 9]$.

($\supseteq$): Given $x \in [0, 9]$, and given $t \in (2, 6)$, note that $2t + 5 > 2(2) + 5 = 9$. Thus, $x \in \mathbb{R}$ with

$$0 \le x \le 9 < 2t + 5, \quad \text{so} \quad x \in [0, 2t + 5).$$

Since this holds for all $t \in (2, 6)$, we have $x \in$LHS. QED

Note: In the ($\subseteq$) step, how did I decide to use contradiction, and how did I think of that choice of t? Well, I carefully wrote out the proof skeleton, knowing that I would be given arbitrary $x \in$LHS and would need to prove $0 \le x \le 9$. The $x \ge 0$ part came easy, so then I needed to prove $x \le 9$. But even though my hypothesis said that for every $t \in (2, 6)$, I couldn't find a (single) value of t for which knowing $x < 2t + 5$ alone would imply $x \le 9$. So after some messing around, I decided it was worth trying a contradiction proof, and seeing what would happen if I assumed $x > 9$. Then I realized I *could* pick a value of $t \in (2, 6)$, chosen to give $x = 2t + 5$ (which is *not* in $[0, 2t + 5)$), by solving $x = 2t + 5$ in my scratchwork to get $t = (x - 5)/2$.

---

2. (10 points) Prove that there is a unique real number $c \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, we have $ct-3c+12 = 4t$.

Proof. (**Existence**): Let $c = 4 \in \mathbb{R}$. Then for any $t \in \mathbb{R}$, we have

$$ct - 3c + 12 = 4t - 12 + 12 = 4t.$$

(**Uniqueness**): Suppose $c_1, c_2 \in \mathbb{R}$ both work for all $t \in \mathbb{R}$. Then in particular, for $t = 0$, we have

$$c_1t - 3c_1 + 12 = 4t = c_2t - 3c_2 + 12, \quad \text{and hence} \quad -3c_1 + 12 = -3c_2 + 12.$$

Subtracting 12 from both sides yields $-3c_1 = -3c_2$, so dividing by -3 gives $c_1 = c_2$. QED

Note 1: How did I think of $c = 4$? By solving the original equation for c in my scratchwork. Rearranging the equation gives $(c-4)(t-3) = 0$. If that's going to be true for *all* $t \in \mathbb{R}$, including t-values other than 3, then we must have $c - 4 = 0$, so $c = 4$.
Note 2: An alternative proof of uniqueness would be to prove that if $c \in \mathbb{R}$ satisfies the equation for all t, then $c = 4$, via factoring the equation as in Note 1 above, and plugging in a specific value for t, like $t = 0$. (Any choice of $t \in \mathbb{R}$ *besides* $t = 3$ would work, but the point is, one needs to actually choose a specific value of t in the proof.)

---

3. (14 points) Prove that for every $y \in [-2, 2]$, there is some $x \in [1, 3]$ such that $\frac{6}{x} - 4 = y$.

Proof. Given $y \in [-2, 2]$, let $x = \dfrac{6}{y+4}$.

Since $y \neq -4$, we have $x \in \mathbb{R}$. In addition, because $y \leq 2$, we have

$$x = \frac{6}{y+4} \geq \frac{6}{2+4} = 1,$$

and because $y \geq -2$, we have

$$x = \frac{6}{y+4} \leq \frac{6}{-2+4} = 3.$$

Thus, $x \in [1, 3]$.

Finally, we have $\dfrac{6}{x} - 4 = \dfrac{6(y+4)}{6} - 4 = y + 4 - 4 = y$. QED

---

4. (11 points) Define a sequence $c_1, c_2, c_3, \ldots$ of real numbers by:

$$c_1 = \frac{1}{2}, \quad \text{and for every } n \geq 1, \quad c_{n+1} = c_n - c_n^2.$$

Use mathematical induction to prove that for all $n \in \mathbb{N}$, we have $0 < c_n < 1$.

Proof. [By induction on $n \geq 1$.]

Base Case: For $n = 1$, we have $0 < \dfrac{1}{2} = c_1 < 1$, as desired.

Inductive Step: Assume the statement is true for some $n = k \geq 1$. Then

$$c_{k+1} = c_k - c_k^2 < c_k < 1,$$

since $c_k^2 > 0$. In addition, we have $c_k > 0$ and $1 - c_k > 0$ by the inductive hypothesis, and hence

$$c_{k+1} = c_k - c_k^2 = c_k(1 - c_k) > 0.$$

Thus, $0 < c_{k+1} < 1$. QED

---

5. (14 points) Use mathematical induction to prove that for all integers $n \geq 2$, we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Proof. [By induction on $n \geq 2$.]

Base Case: For $n = 2$, we have

$$(\text{LHS})^2 = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}\right)^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)^2 = \frac{1 + 2\sqrt{2} + 2}{2} = \frac{3}{2} + \sqrt{2} > 1 + 1 = 2 = (\text{RHS})^2.$$

Taking square roots [and remembering that both LHS and RHS are positive], it follows that LHS > RHS, as desired.

Inductive Step: Assume the statement is true for some $n = k \geq 2$. Observe that $k^2 < k^2 + k = k(k+1)$, and therefore, $k < \sqrt{k(k+1)}$. Hence, $k + 1 < 1 + \sqrt{k} \cdot \sqrt{k+1}$. Dividing both sides by $\sqrt{k+1} > 0$, it follows that $\dfrac{1}{\sqrt{k+1}} + \sqrt{k} > \sqrt{k+1}$. Thus,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}} = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1},$$

where the first inequality is by the inductive hypothesis, and the second is by what we just proved. QED