Solutions to Practice Problems for the Final Exam

1. Let $A = B = \mathbb{R} \setminus \{2\}$, and define $f : A \to B$ by $f(x) = \dfrac{2x+1}{x-2}$.
Decide whether or not f is invertible. If it is, find the inverse function.

Solution/Proof.

$\boxed{\text{Yes, } f \text{ is invertible}}$ and the inverse is $f^{-1} : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{2\}$ by $f^{-1}(x) = \dfrac{2x+1}{x-2}$.

[**Side note**: yes, f and f^{-1} are the same function!]

To prove this, first note that f^{-1} is indeed a function from $\mathbb{R} \setminus \{2\}$ to $\mathbb{R} \setminus \{2\}$, because we have $f^{-1} = f$, and we already know f is a function. [Or, if you want to prove it from scratch: for any $x \in \mathbb{R} \setminus \{2\}$, we have $x \neq 2$, so $f^{-1}(x)$ is defined and equals an element of $\mathbb{R}$. But we do **not** have $f^{-1}(x) = 2$, because if that equality did hold for some such x, that equation implies $2x + 1 = 2(x - 2)$, i.e., $2x + 1 = 2x - 4$, and hence $1 = -4$. So yes indeed, $f^{-1}(x)$ is an element of $\mathbb{R} \setminus \{2\}$.]

It remains to show the two compositions. Since $f^{-1} = f$, we can do both at once. That is, given arbitrary $x \in \mathbb{R} \setminus \{2\}$, we have $f^{-1}(f(x)) = f(f(x)) = f(f^{-1}(x))$, and this common value is

$$f(f(x)) = \frac{2\left(\dfrac{2x+1}{x-2}\right) + 1}{\dfrac{2x+1}{x-2} - 2} = \frac{2(2x+1) + (x-2)}{(2x+1) - 2(x-2)} = \frac{4x + 2 + x - 2}{2x + 1 - 2x + 4} = \frac{5x}{5} = x. \qquad \text{QED}$$

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2. In this problem, you'll prove that $|(0, \infty)| = |[0, \infty)|$ in two different ways.

2a. Write down an explicit function $f : (0, \infty) \to [0, \infty)$ and prove that it is bijective.

2b. Write down (MUCH simpler) functions $g_1 : (0, \infty) \to [0, \infty)$ and $g_2 : [0, \infty) \to (0, \infty)$ and prove that they are injective. Now apply Schröder-Bernstein.

(a): **Answer/Proof**. Define $f : [0, \infty) \to (0, \infty)$ by

$$f(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{Z}, \\ x & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Clearly $f(x) \in (0, \infty)$ for every $x \in [0, \infty)$, so f is actually defined.

To see that f is one-to-one, given $x, y \in [0, \infty)$ with $f(x) = f(y)$. We consider two cases. First, suppose $f(x) \in \mathbb{Z}$. Then $x \in \mathbb{Z}$, as otherwise $f(x) = x \notin \mathbb{Z}$ by definition of f. Similarly, since $f(y) = f(x) \in \mathbb{Z}$, we have $y \in \mathbb{Z}$. Thus,

$$x = (x + 1) - 1 = f(x) - 1 = f(y) - 1 = (y + 1) - 1 = y,$$

as desired. In the second case, suppose $f(x) \notin \mathbb{Z}$. Then $x \notin \mathbb{Z}$, as otherwise we would have $f(x) = x + 1 \in \mathbb{Z}$. Similarly, $y \notin \mathbb{Z}$. Thus, $x = f(x) = f(y) = y$, proving that f is one-to-one.

To see that f is onto, given $y \in (0, \infty)$, we again consider two cases. First, if $y \notin \mathbb{Z}$, then $y \in [0, \infty)$ and $y \notin \mathbb{Z}$, so $f(y) = y$. Second, if $y \in \mathbb{Z}$, then $y \geq 1$, so $y - 1 \in [0, \infty)$. Moreover, $y - 1 \in \mathbb{Z}$, and therefore $f(y - 1) = (y - 1) + 1 = y$, proving that f is onto.

Thus, f is bijective, and hence $|(0, \infty)| = |[0, \infty)|$. QED (a)

(b): **Answer/Proof**. Define $g_1 : (0, \infty) \to [0, \infty)$ by $g_1(x) = x$. Then g_1 is clearly a function. It is also injective, because if $s, t \in (0, 1)$ have $g_1(s) = g_1(t)$, then $s = t$ immediately.

Define $g_2 : [0, \infty) \to (0, \infty)$ by $g_2(x) = x + 1$. Then for any $x \in [0, \infty)$, we have $g_2(x) \geq 1 > 0$, and hence $g_2(x) \in (0, \infty)$. So g_2 is defined, and clearly well-defined. In addition, g_2 is injective, because if $s, t \in [0, \infty)$ have $g_2(s) = g_2(t)$, then $s + 1 = t + 1$, and so $s = t$.

Thus, by Schröder-Bernstein, $|(0,\infty)| = |[0,\infty)|$. QED (b)

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3. Let $n \geq 1$ be an integer, and let $A_1, A_2, \ldots, A_n$ be sets, each of which is countable. Prove that $A_1 \times A_2 \times \cdots \times A_n$ is countable.

Proof, by induction on n. For $n = 1$, we have $A_1 = A_1$ is countable by assumption,
Inductive Step: Assuming the statement is true for some $n \geq 1$, we will prove it for $n+1$. Given $A_1, \ldots, A_{n+1}$ all countable sets, we have

$$A_1 \times \cdots \times A_{n+1} = (A_1 \times \cdots \times A_n) \times A_{n+1}.$$

Now $A_1 \times \cdots \times A_n$ is countable by the inductive hypothesis, and A_{n+1} is countable by assumption. Thus, by a theorem from the book (Corollary 6.3.10) that the product of two countable sets is countable, it follows that $A_1 \times \cdots \times A_{n+1}$ is countable. QED

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4. Let $T = \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $|\mathbb{R}| \neq |T|$.

Proof. Given a function $F : \mathbb{R} \to T$, then for each $s \in \mathbb{R}$, $F(s)$ is itself a function $F(s) : \mathbb{R} \to \mathbb{R}$. So we may define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = 1 + F(x)(x),$$

where $F(x)(x)$ denotes the function $F(x) : \mathbb{R} \to \mathbb{R}$ evaluated at the same point x. Since g is a function $g : \mathbb{R} \to \mathbb{R}$, we have $g \in T$. We claim that g is not in the image of F.
To prove the claim, given $x \in \mathbb{R}$, we need to show that $F(x) \neq g$. We see this by evaluating both functions at x: since

$$g(x) = 1 + F(x)(x) \neq F(x)(x),$$

the functions are indeed different, proving the claim. Thus, F is not onto.
Since is no onto function from $\mathbb{R}$ to T, there is no bijective function, i.e., $|\mathbb{R}| \neq |T|$. QED

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5. Define $f : \mathbb{R} \to (-1,1)$ by $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

5a. Prove that f actually *is* a function from $\mathbb{R}$ to $(-1,1)$.

5b. Prove that f is one-to-one.

Proof. (a): Given $x \in \mathbb{R}$, we have $x^2 + 1 > 0$, so $\sqrt{x^2+1}$ exists and is positive. Thus, $f(x) \in \mathbb{R}$ is indeed a real number. In addition, we have $|x| = \sqrt{x^2} < \sqrt{x^2+1}$, so that $|f(x)| = \dfrac{|x|}{\sqrt{x^2+1}} < 1$, and hence $f(x) \in (-1,1)$. QED (a)

(b): Given $x, y \in \mathbb{R}$ such that $f(x) = f(y)$, we have $\dfrac{x}{\sqrt{x^2+1}} = \dfrac{y}{\sqrt{y^2+1}}$, so that $x\sqrt{y^2+1} = y\sqrt{x^2+1}$.
Squaring both sids, we have $x^2y^2 + x^2 = x^2y^2 + y^2$, and hence $x^2 = y^2$. Thus, $|x| = |y|$.
If $x \geq 0$, then $y = \dfrac{x\sqrt{y^2+1}}{\sqrt{x^2+1}} \geq 0$. Thus, $y = |y| = |x| = x$, as desired.
Otherwise, we have $x < 0$, in which case $y = \dfrac{x\sqrt{y^2+1}}{\sqrt{x^2+1}} < 0$. Then $y = -|y| = -|x| = x$. QED

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6. Use Schröder-Bernstein to prove that:

6a. $|[0,1]| = |\mathbb{R}|$ 6b. $|(0,1]| = |\mathbb{R}|$ 6c. $|\mathbb{R} \smallsetminus \mathbb{Z}| = |\mathbb{R}|$

Proof. In each of the three cases, the set S in question — namely $[0,1]$ in (a), or $(0,1]$ in (b), or $\mathbb{R} \smallsetminus \mathbb{Z}$ in (c) — contains $(0,1)$.

Meanwhile, by Problem #5, there is an injective function $f : \mathbb{R} \to (-1,1)$. Define $g : (-1,1) \to S$ by $g(x) = \frac{1}{2}(x+1)$.

Note that g is defined, and for any $x \in (-1,1)$, we have $g(x) > \frac{1}{2}(-1+1) = 0$, and $g(x) < \frac{1}{2}(1+1) = 1$, so that g is indeed a function $g : (-1,1) \to S$. Moreover, g is injective since if $g(x) = g(y)$, then $\frac{1}{2}(x+1) = \frac{1}{2}(y+1)$, so that $x+1 = y+1$ and hence $x = y$.

Thus, we have an injective function $g \circ f : \mathbb{R} \to S$. We can also define $h : S \to \mathbb{R}$ by $h(x) = x$, which is defined (since any $x \in S$ has $h(x) = x \in \mathbb{R}$) and injective (since if $h(x) = h(y)$, then $x = y$).

In each case, since we have injective functions $g \circ f : \mathbb{R} \to S$ and $h : S \to \mathbb{R}$, then by Schröder-Bernstein, we have $|S| = |\mathbb{R}|$.

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7. Let $S = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$. Prove that:

7a. $S = \big(\mathbb{Z} \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \mathbb{Z}\big)$. 7b. $|\mathbb{Z} \times \mathbb{R}| = |\mathbb{R}|$. 7c. $|S| = |\mathbb{R}|$.

Proof. (a): ($\subseteq$) Given $(x,y) \in S$, if $x \in \mathbb{Z}$, then $(x,y) \in \mathbb{Z} \times \mathbb{R} \subseteq$ RHS. Otherwise, we have $y \in \mathbb{Z}$, and hence $(x,y) \in \mathbb{R} \times \mathbb{Z} \subseteq$ RHS.

($\supseteq$) Given $(x,y) \in$ RHS, we again consider two cases. If $(x,y) \in \mathbb{Z} \times \mathbb{R}$, then $x \in \mathbb{Z}$, and hence $(x,y) \in S$. Otherwise, we have $(x,y) \in \mathbb{R} \times \mathbb{Z}$, so that $y \in \mathbb{Z}$, and hence $(x,y) \in S$. QED

(b,c): **Quick Proof**. [For (c); the quick proof for (b) is similar.] In class, we saw that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Thus, there is a bijective function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Define $f : S \to \mathbb{R} \times \mathbb{R}$ by $f(x,y) = (x,y)$, which is clearly an injective function. Thus, $g \circ f : S \to \mathbb{R}$ is injective.

Define $h : \mathbb{R} \to S$ by $h(x) = (x,0) \in \mathbb{R} \times \mathbb{Z} \subseteq S$. Again, h is clearly an injective function.

Since there are injective functions both ways, we have $|S| = |\mathbb{R}|$, by Schröder-Bernstein. QED

(b): **Longer Proof**. [Without using the powerful fact that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.]

By Problem 5 (or Problem 6), there is an injective function $f : \mathbb{R} \to (0,1)$. Define $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ by $g(m,x) = m + f(x)$. We claim that g is injective. Indeed, given $(m,x),(n,y) \in \mathbb{Z} \times \mathbb{R}$ with $g(m,x) = g(n,y)$, we have $m + f(x) = n + f(y)$. Thus, $f(x) - f(y) = n - m \in \mathbb{Z}$. However, since $f(x), f(y) \in (0,1)$, we have $|f(x) - f(y)| < 1$, and hence $f(x) - f(y)$, being an integer of absolute value less than 1, must be 0. Therefore, we also have $n - m = 0$, i.e., $m = n$. Meanwhile, since $f(x) = f(y)$ and f is injective, we have $x = y$. Hence, $(m,x) = (n,y)$, completing our proof that g is injective.

Meanwhile, define $h : \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$ by $h(x) = (0,x)$. Then h is also injective. After all, given $x, y \in \mathbb{R}$ with $h(x) = h(y)$, we have $(0,x) = (0,y)$, and hence $x = y$.

By Schröder-Bernstein, then, we have $|\mathbb{Z} \times \mathbb{R}| = |\mathbb{R}|$.

(c): **Longer Proof**. [Without using the powerful fact that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.]

By part (b), there's an injective function $F : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, and by #1, there's an injective function $G : \mathbb{R} \to (0,1)$. Define $f : S \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} G(F(x,y)) & \text{if } x \in \mathbb{Z}, \\ G(F(y,x)) + 1 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Note that f is indeed defined. After all, any $(x,y) \in S$ either has $x \in \mathbb{Z}$ or $y \in \mathbb{Z}$. If $x \in \mathbb{Z}$, then $F(x,y)$ is defined and belongs to $\mathbb{R}$, so $G(F(x,y)) \in (0,1) \subseteq \mathbb{R}$. Otherwise, i.e., if $x \notin \mathbb{Z}$, then we must have $y \in \mathbb{Z}$, and hence $F(y,x)$ is defined, so $G(F(y,x)) \in (0,1)$, and hence $f(x,y) \in (1,2) \subseteq \mathbb{R}$.

[The idea is that f maps the vertical lines in S into $(0,1)$, and it maps the horizontal lines — at least the portions that don't intersect the vertical lines — into $(1,2)$.]

Then f is also injective, as we now prove. Given $(s,t),(x,y) \in S$ with $f(s,y) = f(x,y)$, this common value lies in either $(0,1)$ or $(1,2)$. If it lies in $(1,2)$, then we have $G(F(t,s)) + 1 = G(F(y,x)) + 1$, so $G(F(t,s)) = G(F(y,x))$. Since G and F are injective, we have $(t,s) = (y,x)$, so that $(s,t) = (x,y)$.

Similarly, if the common value lies in $(0,1)$, then $G(F(t,s)) = G(F(y,x))$, which gives $(s,t) = (x,y)$ by the same argument, proving our claim.
Next, define $h : \mathbb{R} \to S$ by $h(x) = (0,x)$. Then h is also injective, as in part (b): given $x, y \in \mathbb{R}$ with $h(x) = h(y)$, we have $(0,x) = (0,y)$, and hence $x = y$.
By Schröder-Bernstein, then, we have $|S| = |\mathbb{R}|$.

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8. Prove that $\displaystyle\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 1+\frac{3}{n}\right] = (0,4]$

Proof. ($\subseteq$): Givens $x \in$ LHS, there exists $n \in \mathbb{N}$ such that $x \in \left[\frac{1}{n}, 1+\frac{3}{n}\right]$. Thus, we have

$$0 < \frac{1}{n} \le x \le 1 + \frac{3}{n} \le 4,$$

and hence $x \in (0,4]$. QED ($\subseteq$)

($\supseteq$): Given $x \in (0,4]$, so that $0 < x \le 4$, we consider two cases.
If $x \ge 1$, then $x \in [1,4] \subseteq$ LHS, since $[1,4]$ is the interval in the union for $n = 1$.
Otherwise, we have $0 < x < 1$. Pick $n \in \mathbb{N}$ such that $n > \frac{1}{x}$. [For example, we could pick $n = \lceil \frac{1}{x} \rceil + 1$.]
Then $\frac{1}{n} < x < 1 \le 1 + \frac{3}{n}$, and hence $x \in \left[\frac{1}{n}, 1+\frac{3}{n}\right] \subseteq$ LHS. QED ($\supseteq$)

QED

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9. Prove that $\displaystyle\bigcap_{n\in\mathbb{N}} \left[\frac{1}{n}, 1+\frac{3}{n}\right] = \{1\}$

Proof. ($\subseteq$): Given $x \in$ LHS, then we have, using $n = 1$, that $x \in [1,4]$, so $x \ge 1$.
Suppose, towards a contradiction, that $x > 1$. Pick $n \in \mathbb{N}$ such that $n > \frac{3}{x-1}$. [For example, we could pick $n = \left\lceil \frac{3}{x-1} \right\rceil + 1$.] Then since $n, x-1 > 0$, we have $x - 1 > \frac{3}{n}$, and hence $x > 1 + \frac{3}{n}$. It follows that $x \notin \left[\frac{1}{n}, 1+\frac{3}{n}\right]$, and hence $x \notin$ LHS, a contradiction. Therefore, we must have $x \le 1$.
Since $x \ge 1$ and $x \le 1$, we have $x = 1 \in \{1\}$. QED ($\subseteq$)

($\supseteq$): Given $x \in \{1\}$, then given any $n \in \mathbb{N}$, we have

$$\frac{1}{n} \le 1 = x \le 1 + \frac{3}{n},$$

so $x \in \left[\frac{1}{n}, 1+\frac{3}{n}\right]$. Thus, $x \in$ LHS. QED ($\supseteq$)

QED

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10. Prove, from the ε-N definition, that $\displaystyle\lim_{n\to\infty} \frac{6n^2-7}{n^2+1} = 6$

Proof. Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $N > 13/\varepsilon$. Given $n \ge N$, we have $N \le n \le n^2 < n^2+1$, so

$$\left|\frac{6n^2-7}{n^2+1} - 6\right| = \left|\frac{6n^2-7-6n^2-6}{n^2+1}\right| = \left|\frac{-13}{n^2+1}\right| = \frac{13}{n^2+1} < \frac{13}{N} < \frac{13}{13/\varepsilon} = \varepsilon$$ QED

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11. Prove, from the ε-N definition, that $\displaystyle\lim_{n\to\infty} \frac{3+7n^2-6n^3}{n^3-4n} = -6$

Proof. Given $\varepsilon > 0$, pick $N_1 \in \mathbb{N}$ such that $N_1 > 14/\varepsilon$, and define $N = \max\{4, N_1\}$.

Given $n \geq N$, then $n^3 \geq 4^2 \cdot n > 8n$, so that $n^3 - 4n = \frac{n^3}{2} + \frac{n^3}{2} - 4n > \frac{n^3}{2}$. We also have $7n^2 - 24n + 3 > 6n(n-4) > 0$ and furthermore $7n^2 - 24n + 3 < 7n^2$. Therefore,

$$\left|\frac{3 + 7n^2 - 6n^3}{n^3 - 4n} - (-6)\right| = \left|\frac{7n^2 - 24n + 3}{n^3 - 4n}\right| = \frac{7n^2 - 24n + 3}{n^3 - 4n} < \frac{7n^2}{n^3/2} = \frac{14}{n} \leq \frac{14}{N_1} < \frac{14}{14/\varepsilon} = \varepsilon$$

QED

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12. For each of the following sequences, decide whether it converges, diverges to ∞, diverges to $-\infty$, or diverges but not to either ∞ or $-\infty$. (And prove your claims, of course.)

12a. $\left(\frac{5n^3 + 7n}{2n^3 - 11}\right)_{n=1}^{\infty}$ 12b. $\left(\frac{2n^2 - 55}{40n + 100}\right)_{n=1}^{\infty}$

12c. $\left(\frac{3^{n+2} + 7}{3^n - 2}\right)_{n=1}^{\infty}$ 12d. $\left(7n + (-1)^n \cdot n^2\right)_{n=1}^{\infty}$

Proof. (a): We claim $\boxed{\text{the sequence converges to } \frac{5}{2}}$

Given $\varepsilon > 0$, pick $N_1 \in \mathbb{N}$ such that $N_1 \geq \frac{8}{\varepsilon}$, and let $N = \max\{N_1, 30\} \in \mathbb{N}$.
Given $n \geq N$, we have $14n + 55 = 16n + (55 - 2n) \leq 16n + 55 - 2(30) < 16n$,
and $2n^3 - 11 = n^3 + (n^3 - 11) \geq n^2 + 0 = n^2$. Thus,

$$\left|\frac{5n^3 + 7n}{2n^3 - 11} - \frac{5}{2}\right| = \left|\frac{2(5n^3 + 7n) - 5(2n^3 - 11)}{2(2n^3 - 11)}\right| = \frac{14n + 55}{2(2n^3 - 11)} < \frac{16n}{2n^2} = \frac{8}{n} \leq \frac{8}{8/\varepsilon} = \varepsilon$$

QED (a)

(b): We claim $\boxed{\text{the sequence diverges to } \infty}$
Given $M > 0$, pick $N_1 \in \mathbb{N}$ with $N_1 > 42M$, and let $N = \max\{N_1, 55\} \in \mathbb{N}$.
Given $n \geq N$, we have $2n^2 - 55 \geq n^2 + n - 55 \geq n^2$, and $40n + 100 = 42n + 2(50 - n) < 42n$. Thus,

$$\frac{2n^2 - 55}{40n + 100} > \frac{n^2}{42n} = \frac{n}{42} \geq \frac{N}{42} > M$$

QED (b)

(c): We claim $\boxed{\text{the sequence converges to } 9}$
Given $\varepsilon > 0$, pick $N_1 \in \mathbb{N}$ such that $N_1 \geq \frac{50}{\varepsilon}$, and let $N = \max\{N_1, 2\} \in \mathbb{N}$.
Given $n \geq N$, we have $3^n \geq 3^2 = 9$, so $3^n - 2 = \frac{3^n}{2} + \frac{3^n - 4}{2} > \frac{3^n}{2}$. We also have $3^n > n$, so

$$\left|\frac{3^{n+2} + 7}{3^n - 2} - 9\right| = \left|\frac{9 \cdot 3^n + 7 - 9 \cdot 3^n - 9(-2)}{3^n - 2}\right| = \frac{25}{3^n - 2} < \frac{50}{3^n} < \frac{50}{n} \leq \frac{50}{50/\varepsilon} = \varepsilon$$

QED (c)

(d): We claim $\boxed{\text{the sequence diverges, and not even to } \pm\infty}$
To see this, write the elements of the sequence as $a_n = 7n + (-1)^n \cdot n^2$.
Let (b_n) be the subsequence given by $b_n = a_{2n} = 14n + 4n^2$. We will show that (b_n) diverges to ∞. To see this, given $M > 0$, pick $N \in \mathbb{N}$ with $N > M$. Then for any $n \geq N$, we have

$$b_n = 14n + 4n^2 > n \geq N > M,$$

proving our claim that $\lim_{n\to\infty} b_n = \infty$. Thus, the original sequence (a_n) diverges, because if it converged, then all of its subsequences would also converge (to the same limit), but the subsequence (b_n) does not.

We also have that (a_n) does not diverge to $-\infty$, because choosing $M = 1$ (and hence $-M = -1$), for any $N \in \mathbb{N}$, we may select $n \geq N$ even, so that $a_n = 7n + n^2 \geq 0$, so that $a_n \not< -1$.

Finally, (a_n) also does not diverge to ∞, because choosing $M = 1$, for any $N \in \mathbb{N}$, we may select $n \geq \max\{N, 8\}$ odd so that $7 - n \leq -1$, and hence $a_n = 7n - n^2 = n(7 - n) \leq -n < 0$, so that $a_n \not> 1$.

QED (d)

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13. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are real sequences such that $(a_n)_{n=1}^{\infty}$ is bounded and $\lim\limits_{n\to\infty} b_n = 0$. Prove that $\lim\limits_{n\to\infty} a_n \cdot b_n = 0$.

Proof. Let $M \in \mathbb{R}$ be a bound for (a_n), so that for every $n \in \mathbb{N}$, we have $|a_n| \le M$. Increasing M if necessary, we may assume that $M > 0$.

Given $\varepsilon > 0$, then because $\lim\limits_{n\to\infty} b_n = 0$, there is some $N \in \mathbb{N}$ such that for all $n \ge N$, we have $|b_n - 0| < \dfrac{\varepsilon}{M}$.

Then for any $n \ge N$, we have

$$|a_n b_n - 0| = |a_n| \cdot |b_n| \le M \cdot |b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

QED

Note: Alternatively, after choosing the bound M, we can define two sequence (c_n) and (d_n), given by $c_n = M|b_n|$ and $d_n = -M|b_n|$. It is then not difficult to prove that $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} d_n = 0$, and also that $d_n \le a_n \cdot b_n \le c_n$ for every $n \in \mathbb{N}$. Thus, by the Squeeze Theorem, the desired result follows.

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14. Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent real sequence. Prove that $(a_n)_{n=1}^{\infty}$ is bounded.

Proof. Let $L = \lim\limits_{n\to\infty} a_n \in \mathbb{R}$. Choosing $\varepsilon = 1 > 0$, there is some $N \in \mathbb{N}$ such that for all $n \ge N$, we have $|a_n - L| < 1$.

Let $M_1 = \max\{|a_1|, |a_2|, \dots, |a_N|\}$, and let $M = \max\{M_1, 1 + |L|\} > 0$. We claim that M is a bound for the sequence (a_n).

To see this, given $n \in \mathbb{N}$, there are two cases. If $n < N$, then $|a_n| \le M_1 \le M$. Otherwise, we have $n \ge N$, in which case

$$|a_n| = |a_n - L + L| \le |a_n - L| + |L| < 1 + |L| \le M.$$

QED

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15. Define a real sequence $(a_n)_{n=1}^{\infty}$ by

$$a_1 = 0, \qquad \text{and for all } n \in \mathbb{N}, \quad a_{n+1} = a_n^2 + \frac{1}{4}.$$

In this problem, you'll prove that $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$, via the following steps:

15a. Prove that for every $n \in \mathbb{N}$, we have $a_{n+1} \ge a_n$.

15b. Use induction to prove that for every $n \in \mathbb{N}$, we have $0 \le a_n < \dfrac{1}{2}$.

15c. Prove that $\lim\limits_{n\to\infty} a_n$ converges to some number $L \in \mathbb{R}$

15d. Justify each $=$ sign in the following: $L^2 + \dfrac{1}{4} = \lim\limits_{n\to\infty} a_n^2 + \dfrac{1}{4} = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$

15e. Conclude that $L = \dfrac{1}{2}$.

Proof. (a): Given $n \ge \mathbb{N}$, we have

$$a_{n+1} - a_n = a_n^2 - a_n + \frac{1}{4} = \left(a_n - \frac{1}{2}\right)^2 \ge 0.$$

Thus, $a_{n+1} \ge a_n$. QED (a)

(b): By induction on $n \ge 1$:

Base Case: We have $a_1 = 0 < \dfrac{1}{2}$ QED Base

Inductive Step: Assume it is true for some $n = k$. By (a) we have $a_{k+1} \geq a_k \geq 0$, proving the first inequality.

We also have $a_{k+1} = a_k^2 + \frac{1}{4} < \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, proving the second inequality. QED (b)

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(c): By parts (a) and (b), the sequence (a_n) is increasing and bounded above, so by the Monotone Sequence Theorem, it converges to some $L \in \mathbb{R}$. QED (c)

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(d): By the limit laws, we have

$$\lim_{n\to\infty}\left(a_n^2 + \frac{1}{4}\right) = \lim_{n\to\infty} a_n \cdot a_n + \lim_{n\to\infty}\frac{1}{4} = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} a_n\right) + \frac{1}{4} = L^2 + \frac{1}{4},$$

verifying the first $=$ sign.

By hypothesis, we have $a_{n+1} = a_n^2 + \frac{1}{4}$, and hence $\lim_{n\to\infty} a_n^2 + \frac{1}{4} = \lim_{n\to\infty} a_{n+1}$, verifying the second $=$ sign.

The sequence $(a_{n+1})_{n=1}^{\infty} = (a_n)_{n=2}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. Since $(a_n)_{n=1}^{\infty}$ converges to L, the subsequence converges to the same limit L, justifying the third and fourth $=$ signs. QED (d)

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(e): Since $L^2 + \frac{1}{4} = L$, we have $L^2 - L + \frac{1}{4} = 0$, i.e. $\left(L - \frac{1}{2}\right)^2 = 0$, so $L - \frac{1}{2} = 0$, and hence $L = \frac{1}{2}$.
QED (e)

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16. Define a real sequence $(b_n)_{n=1}^{\infty}$ by

$$b_1 = 1, \qquad \text{and for all } n \in \mathbb{N}, \quad b_{n+1} = b_n^2 + \frac{1}{4}.$$

Prove that $\lim_{n\to\infty} b_n$ diverges to ∞.

Proof. We claim that for all $n \geq 4$, we have $b_n \geq n - 1$. We prove this claim by induction on $n \geq 4$.

Base Case: We have $b_1 = 1$, so $b_2 = 1^2 + \frac{1}{4} = \frac{5}{4}$, so $b_3 = \frac{25}{16} + \frac{1}{4} = \frac{29}{16} > \frac{7}{4}$.

Thus, $b_4 > \frac{49}{16} + \frac{1}{4} = \frac{53}{16} > 3 = 4 - 1$. QED Base

Inductive Step: Suppose the claim holds for some $n = k \geq 4$.

Then $b_{k+1} = b_k^2 + \frac{1}{4} > (k-1)^2 + \frac{1}{4} > k$, where the last equality is because

$(k-1)^2 + \frac{1}{4} - k = k^2 - 2k + 1 + \frac{1}{4} - k > k^2 - 3k = k(k-3) > 0$,

since $k > 3$. QED Claim

To prove that $\lim_{n\to\infty} b_n = \infty$, given $M > 0$, pick $N_1 \in \mathbb{N}$ such that $N_1 > M + 1$. Let $N = \max\{N_1, 4\}$.

Given $n \geq N$, we have $b_n \geq n - 1 \geq N - 1 > (M + 1) - 1 = M$,

where the first inequality is by the claim, since $n \geq 4$. QED

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17. Define a real sequence $(c_n)_{n=1}^{\infty}$ by

$$c_1 = 2, \qquad \text{and for all } n \in \mathbb{N}, \quad c_{n+1} = \frac{3c_n}{4} + \frac{3}{c_n}.$$

Follow a similar strategy as in Problem 15 to prove that $\lim_{n\to\infty} c_n$ converges and equals $2\sqrt{3}$.

Proof. First, we claim that for every $n \in \mathbb{N}$, we have $2 \leq c_n \leq 2\sqrt{3}$. We proceed by induction on $n \geq 1$.

Base Case: For $n = 1$, we have $2 = c_1 \leq 2\sqrt{3}$ by definition. QED Base

Inductive Step: Assume our claim holds for some $n = k \geq 1$.

We have $\frac{3}{4}c_k^2 - 2\sqrt{3}c_k + 3 = \frac{3}{4}\left(c_k - \frac{2}{\sqrt{3}}\right)(c_k - 2\sqrt{3}) \leq 0$, since $\frac{2}{\sqrt{3}} < 2 \leq c_k \leq 2\sqrt{3}$.
Therefore $c_{k+1} = \frac{3c_k}{4} + \frac{3}{c_k} = \frac{1}{c_k}\left(\frac{3}{4}c_k^2 + 3\right) \leq \frac{1}{c_k}(2\sqrt{3}c_k) = 2\sqrt{3}$. QED Claim

Second, we further claim that for all $n \in \mathbb{N}$, we have $c_n \leq c_{n+1}$.
To see this, given any $n \in \mathbb{N}$, by our first claim we have $c_n^2 \leq (2\sqrt{3})^2 = 12$.
Since we also have $c_n \geq 2 > 0$, dividing by $4c_n$ yields $\frac{c_n}{4} \leq \frac{3}{c_n}$,
and hence $c_n \leq \frac{3c_n}{4} + \frac{3}{c_n} = c_{n+1}$, proving our second claim.

Thus, (c_n) is an increasing sequence that is bounded above, and hence it converges by the Monotone Sequence Theorem to some limit $L \in \mathbb{R}$.

Because $(c_{n+1})_{n=1}^{\infty}$ is a subsequence of (c_n), we also have $\lim_{n\to\infty} c_{n+1} = c_n$.
By the limit laws, it follows that

$$L^2 = L \cdot L = \lim_{n\to\infty} c_n \cdot \lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} (c_n \cdot c_{n+1})$$
$$= \lim_{n\to\infty} c_n \left(\frac{3c_n}{4} + \frac{3}{c_n}\right) = \lim_{n\to\infty} \left(\frac{3}{4}c_n^2 + 3\right) = \frac{3}{4}L^2 + 3.$$

Rearranging, we have $\frac{1}{4}L^2 = 3$, so that $L^2 = 12$ and hence $L = \pm\sqrt{12}$. But because $c_n > 0$ for all n, we must have $L = \sqrt{12} = 2\sqrt{3}$. QED