Math 220, Section 03, Fall 2025 Professor Rob Benedetto

Practice Problems for the Final Exam

Warning: The final exam is comprehensive, but problems on this review sheet are mostly on material from Sections 6.3, 6.4, 8.3, and 8.4. (Well, there are also a few problems from old material.) So use *all three* practice problem sheets as you study for the final. As before, the problems here are, on average, more difficult than those on the exam itself will be.

1. Let $A = B = \mathbb{R} \smallsetminus \{2\}$, and define $f : A \to B$ by $f(x) = \dfrac{2x+1}{x-2}$.
Decide whether or not f is invertible. If it is, find the inverse function.
(And as always, make sure to prove all of your claims.)

2. In this problem, you'll prove that $|(0,\infty)| = |[0,\infty)|$ in two different ways.

   2a. Write down an explicit function $f : (0,\infty) \to [0,\infty)$ and prove that it is bijective.
   (*Hint*: you'll have to use some kind of piecewise-defined function here.)

   2b. Write down (MUCH simpler) functions $g_1 : (0,\infty) \to [0,\infty)$ and $g_2 : [0,\infty) \to (0,\infty)$ and prove that they are injective. Now apply Schröder-Bernstein.

3. Let $n \geq 1$ be an integer, and let $A_1, A_2, \ldots, A_n$ be sets, each of which is countable. Prove that $A_1 \times A_2 \times \cdots \times A_n$ is countable.
(*Suggestion*: Use Corollary 6.3.10, that $A_1 \times A_2$ is countable, and induction on n.)

4. Let $T = \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $|\mathbb{R}| \neq |T|$.
(*Hint*: Given any function $F : \mathbb{R} \to T$, show that F can't be onto by constructing an element of T that F cannot possibly hit.)

5. Define $f : \mathbb{R} \to (-1,1)$ by $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

   5a. Prove that f actually *is* a function from $\mathbb{R}$ to $(-1,1)$.

   5b. Prove that f is one-to-one.

[*Note*: In fact, f is also onto; but I am only asking you to prove that it is a one-to-one function.]

6. Use Schröder-Bernstein to prove that:

   6a. $|[0,1]| = |\mathbb{R}|$ 6b. $|(0,1]| = |\mathbb{R}|$ 6c. $|\mathbb{R} \smallsetminus \mathbb{Z}| = |\mathbb{R}|$

[*Suggestion*: use Problem 5.]

7. Let $S = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.

   7a. Prove that $S = \big(\mathbb{Z} \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \mathbb{Z}\big)$.

   7b. Prove that $|\mathbb{Z} \times \mathbb{R}| = |\mathbb{R}|$. (*Suggestion*: use Schröder-Bernstein.)

   7c. Prove that $|S| = |\mathbb{R}|$.
   (*Suggestion*: See if you can do it using 5b and some extra work.
   There's also a quick way using the fact that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.
   Either way, you'll almost certainly need Schröder-Bernstein.)

8. Prove that $\displaystyle\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1 + \frac{3}{n}\right] = (0,4]$

9. Prove that $\bigcap_{n\in\mathbb{N}} \left[\frac{1}{n}, 1+\frac{3}{n}\right] = \{1\}$

10. Prove, from the ε-N definition, that $\lim_{n\to\infty} \frac{6n^2-7}{n^2+1} = 6$

11. Prove, from the ε-N definition, that $\lim_{n\to\infty} \frac{3+7n^2-6n^3}{n^3-4n} = -6$

12. For each of the following sequences, decide whether it converges, diverges to ∞, diverges to $-\infty$, or diverges but not to either ∞ or $-\infty$. (And prove your claims, of course.)

12a. $\left(\frac{5n^3+7n}{2n^3-11}\right)_{n=1}^{\infty}$ 12b. $\left(\frac{2n^2-55}{40n+100}\right)_{n=1}^{\infty}$

12c. $\left(\frac{3^{n+2}+7}{3^n-2}\right)_{n=1}^{\infty}$ 12d. $\left(7n+(-1)^n\cdot n^2\right)_{n=1}^{\infty}$

13. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are real sequences such that $(a_n)_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} b_n = 0$. Prove that $\lim_{n\to\infty} a_n \cdot b_n = 0$.

14. Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent real sequence. Prove that $(a_n)_{n=1}^{\infty}$ is bounded.

15. Define a real sequence $(a_n)_{n=1}^{\infty}$ by

$$a_1 = 0, \qquad \text{and for all } n\in\mathbb{N}, \quad a_{n+1} = a_n^2 + \frac{1}{4}.$$

In this problem, you'll prove that $\lim_{n\to\infty} a_n = \frac{1}{2}$, via the following steps:

15a. Prove that for every $n\in\mathbb{N}$, we have $a_{n+1} \geq a_n$.

15b. Use induction to prove that for every $n\in\mathbb{N}$, we have $0 \leq a_n < \frac{1}{2}$.

15c. Prove that $\lim_{n\to\infty} a_n$ converges to some number $L\in\mathbb{R}$

15d. Justify each $=$ sign in the following: $L^2 + \frac{1}{4} = \lim_{n\to\infty} a_n^2 + \frac{1}{4} = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$

15e. Conclude that $L = \frac{1}{2}$.

16. Define a real sequence $(b_n)_{n=1}^{\infty}$ by

$$b_1 = 1, \qquad \text{and for all } n\in\mathbb{N}, \quad b_{n+1} = b_n^2 + \frac{1}{4}.$$

(Note the similarity to the sequence in the previous problem; but we are starting with $b_1 = 1$.) Prove that $\lim_{n\to\infty} b_n$ diverges to ∞.

(*Suggestion*: First prove that for every $n \geq 4$, we have $b_n \geq n-1$.)

17. Define a real sequence $(c_n)_{n=1}^{\infty}$ by

$$c_1 = 2, \qquad \text{and for all } n\in\mathbb{N}, \quad c_{n+1} = \frac{3c_n}{4} + \frac{3}{c_n}.$$

Follow a similar strategy as in Problem 15 to prove that $\lim_{n\to\infty} c_n$ converges and equals $2\sqrt{3}$.

(*Suggestion*: Prove that for every $n\in\mathbb{N}$, we have $2 \leq c_n \leq 2\sqrt{3}$ and $c_n \leq c_{n+1}$.)