Solutions to Practice Problems 2

1. Prove or disprove the following statement: for all integers $m, n \in \mathbb{Z}$, if $m|n^2$, then $m|n$.

(Dis)Proof. This is false. Let $m = 4$ and $n = 2$. Then $m = 4|4 = n^2$, but $m = 4 \nmid 2 = n$. QED

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2. Suppose that $a, b, c \in \mathbb{Z}$ are nonzero integers such that $143a + 217b = c$. Prove that $\gcd(a, b)|c$.

Proof. Write $d = \gcd(a, b)$. Since $d|a$ and $d|b$, there are integers $k, m \in \mathbb{Z}$ such that $a = kd$ and $b = md$. Thus,

$$c = 143a + 217b = 143kd + 217md = (143k + 217m)d.$$

Since $143k + 217m \in \mathbb{Z}$, we have $d|c$. QED

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3. Prove that $4|(13^n - 1)$ for every $n \in \mathbb{N}$.

Proof, by induction on n.

Base Case: For $n = 1$, we have $13^n - 1 = 12 = 4 \cdot 3$, which is divisible by 4.

Inductive Step: Given that it's true for $n = k \geq 1$, we may write $13^k - 1 = 4m$ for some $m \in \mathbb{Z}$, and hence

$$13^{k+1} - 1 = 13(13^k - 1) + 13 - 1 = 13(4m) + 12 = 4(13m + 3).$$

Since $13m + 3 \in \mathbb{Z}$, we have $4|(13^{k+1} - 1)$. QED

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4. Prove that $7|(10^n - 3^n)$ for every $n \in \mathbb{N}$.

Proof, by induction on n.

Base Case: For $n = 1$, we have $10^n - 3^n = 10 - 3 = 7 = 7 \cdot 1$, which is divisible by 7.

Inductive Step: Given that it's true for $n = k \geq 1$, we may write $10^k - 3^k = 7m$ for some $m \in \mathbb{Z}$, and hence

$$10^{k+1} - 3^{k+1} = 10(10^k - 3^k) + 10 \cdot 3^k - 3^{k+1} = 10(7m) + (10 - 3) \cdot 3^k = 7(10m + 3^k).$$

Since $10m + 3^k \in \mathbb{Z}$, we have $7|(10^{k+1} - 3^{k+1})$. QED

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5. Let $n \in \mathbb{N}$ be a positive integer, and write its prime factorization as $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, and $r_1, r_2, \ldots, r_k \in \mathbb{N}$. Prove that n is a perfect cube if and only if each of the integers $r_1, r_2, \ldots, r_k$ is divisible by 3.

Proof. ($\Rightarrow$): Given that n is a perfect cube, there exists $m \in \mathbb{N}$ such that $m^3 = n$. Write the prime factorization of m as $m = q_1^{s_1} \cdots q_\ell^{s_\ell}$, where $q_1, \ldots, q_\ell$ are distinct primes, and each $s_i \in \mathbb{N}$. Then

$$p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = n = m^3 = q_1^{3s_1} \cdots q_\ell^{3s_\ell}.$$

Thus, possibly after re-ordering, the q_i's are precisely the p_i's, and $r_i = 3s_i$ for each i. Thus, each r_i is divisible by 3.

($\Leftarrow$): Given that each r_i is divisible by 3, there are integers $s_1, \ldots, s_k \in \mathbb{N}$ such that $r_i = 3s_i$ for each i. Define

$$m = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \in \mathbb{N}.$$

Then

$$m^3 = p_1^{3s_1} p_2^{3s_2} \cdots p_k^{3s_k} = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = n,$$

so that $n = m^3$ is a perfect cube. QED

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6. Let $a, b, c \in \mathbb{N}$ be positive integers. Suppose that $c = a^2$, but also $c = b^3$. Prove that there is some $n \in \mathbb{N}$ such that $c = n^6$.

Proof. Write the prime factorization of c as $c = p_1^{r_1} \cdots p_k^{r_k}$, where $p_1, \ldots, p_k$ are distinct primes, and each $r_i \in \mathbb{N}$. By the previous problem, the fact that c is a perfect cube means that $3|r_i$ for each i. A similar argument shows, because c is a perfect square, that $2|r_i$ for each i. Thus, for each i, the prime factorization of r_i has (at least) a 2 and a 3 in it, meaning that $r_i = 6s_i$ for some $s_i \in \mathbb{N}$. Define

$$n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \in \mathbb{N}.$$

Then $n^6 = p_1^{6s_1} p_2^{6s_2} \cdots p_k^{6s_k} = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = c$. QED

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7. Let $p, q \in \mathbb{N}$ be prime numbers. Suppose that $p|q$. Prove that $p = q$.

Proof. Since q is prime, the only elements of $\mathbb{N}$ dividing q are 1 and q. Thus, since $p \in \mathbb{N}$ divides q, we have either $p = 1$ or $p = q$. However, $p \geq 2$ since p is prime. Thus, $p = q$. QED

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8. Let $m, n \in \mathbb{Z}$ be positive integers. Suppose that $20|(m-7)$ and $20|(n-11)$. Prove that $20|(mn-17)$.

Proof. There exist $a, b \in \mathbb{Z}$ such that $m - 7 = 20a$ and $n - 11 = 20b$. That is, $m = 7 + 20a$ and $n = 11 + 20b$. Thus,

$$mn - 17 = (7 + 20a)(11 + 20b) - 17 = 60 + 20(11a) + 20(7b) + 20(20ab) = 20(3 + 11a + 7b + 20ab).$$

Since $3 + 11a + 7b + 20ab \in \mathbb{Z}$, we have $20|(mn - 17)$. QED

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9. Let $a, b, c \in \mathbb{Z}$, and suppose that $a|(15b + 31c)$ and that $a|15$. Prove that $a|c$.

Proof. By hypothesis, there exist integers $m, n \in \mathbb{Z}$ such that $15b + 31c = ma$ and $15 = na$. The first equation gives us $31c = ma - 15b$, and hence $c = ma - 15b - 15(2c)$. Thus,

$$c = ma - 15b - 15(2c) = ma - nab - 2nac = a(m - nb - 2nc).$$

Since $m - nb - 2nc \in \mathbb{Z}$, it follows that $a|c$. QED

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10. Let $a, b, m, n \in \mathbb{N}$, and suppose that $am = bn$ and that $\gcd(a, b) = 1$. Prove that there is some $k \in \mathbb{N}$ such that $n = ka$ and $m = kb$.

Proof (Method 1). Write $a = p_1^{r_1} \cdots p_t^{r_t}$, where $p_1, \ldots, p_t$ are distinct primes, and each $r_i \in \mathbb{N}$. Similarly write $b = q_1^{s_1} \cdots q_\ell^{s_\ell}$, where $q_1, \ldots, q_\ell$ are distinct primes, and each $s_i \in \mathbb{N}$.
Note that for each i, j, we have $p_i \neq q_j$, as otherwise p_i would be a common divisor of a and b, so that $\gcd(a, b) \geq p_i > 1$, a contradiction. Thus, the primes $p_1, \ldots, p_t, q_1, \ldots, q_\ell$ are all distinct.
Let $N = am = bn$. For each $i = 1, \ldots, t$, we have $p_i^{r_i}|a$, and hence $p_i^{r_i}|N$. Similarly, for each $j = 1, \ldots, \ell$, we have $q_j^{s_j}|b$, and hence $q_j^{s_j}|N$. Thus, the prime factorization of N includes each p_i to at least the power r_i, and each q_j to at least the power s_j. Since these primes are all distinct, it follows that N is divisible by $p_1^{r_1} \cdots p_t^{r_t} q_1^{s_1} \cdots q_\ell^{s_\ell} = ab$.
That is, there is an integer $k \in \mathbb{N}$ such that $N = kab$. Thus, $am = N = kab$, and hence, dividing by a, we have $m = kb$. Similarly, we have $bn = N = kab$, and hence $n = kb$. QED

Proof (Method 2). Since $am = bn$, we have $a|(bn)$. But then, because $\gcd(a, b) = 1$, it follows that $a|n$, by a theorem (e.g. from the book). That is, there exists $k \in \mathbb{Z}$ such that $n = ka$. Since $a, n > 0$, we must have $k > 0$, so $k \in \mathbb{N}$.
Thus, we have $am = bka$, and cancelling a from both sides, we have $m = bk$. So $n = ka$ and $m = kb$, QED

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Note: here's a quick proof of the theorem quoted in Method 2 above. Since $\gcd(a,b) = 1$, there exist integers $x, y \in \mathbb{Z}$ such that $xa + yb = 1$. Multiplying by n, we have $xan + ybn = n$.
But $bn = am$, so $xan + yam = n$. That is, $a(xn + ym) = n$. Define $k = xn + ym \in \mathbb{Z}$; then $ak = n$, i.e., $a|n$.

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11. Determine whether each of the following supposed functions is actually a function.

(a) $f : (0, \infty) \to (0, \infty)$ by $f(x) = \dfrac{x-1}{\lceil x \rceil}$ [$\lceil x \rceil$ denotes the ceiling function of x]

(b) $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by $g(A) = \{n + 3 \,|\, n \in A\}$.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ by $h(A) = \min A$. [$\min A$ denotes the smallest element of A.]

(d) $k : \mathcal{P}(\mathbb{N}) \smallsetminus \{\varnothing\} \to \mathbb{N}$ by $k(A) = \max A$. [$\max A$ denotes the largest element of A.]

(e) $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ by $F(n) = \{m \in \mathbb{N} \,|\, m \text{ is a divisor of } n\}$.

Answers/Proofs. (a): Not a function. $\lceil 1/2 \rceil = 1$, so

$$f\Big(\frac{1}{2}\Big) = \frac{-1/2}{1} = -\frac{1}{2} \notin (0, \infty),$$

so f does not actually map $(0, \infty)$ into $(0, \infty)$.
[Note: if we had written $f : (0, \infty) \to (-1, \infty)$, that *would* have been a funciton.]

(b): Function. Given $A \in \mathcal{P}(\mathbb{N})$, we have $A \subseteq \mathbb{N}$. So for all $n \in A$, we have $n + 3 \in \mathbb{N}$. Thus, $g(A)$ is indeed a defined and well-defined subset of $\mathbb{N}$, and hence $g(A) \in \mathcal{P}(\mathbb{N})$.

(c): Not a function. $\varnothing \in \mathcal{P}(\mathbb{N})$, but $\varnothing$ has no elements at all, and hence no smallest element. Thus, $h(\varnothing)$ isn't defined.

(d): Not a function. $\mathbb{N} \in \mathcal{P}(\mathbb{N}) \smallsetminus \{\varnothing\}$, but $\mathbb{N}$ has no largest element, so $k(\mathbb{N})$ isn't defined.

(e): Function. For each $n \in \mathbb{N}$, and for each $m \in \mathbb{N}$, the statement "m is a divisor of n" is indeed a statement (i.e., either true or false but not both), and hence $F(n)$ is indeed a set, and in fact a subset of $\mathbb{N}$. That is, $F(n)$ is a defined and well-defined element of $\mathcal{P}(\mathbb{N})$.

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12. For each of the following functions, decide whether or not it is injective, and also whether or not it is surjective. If it is both, find a formula for its inverse function. Don't forget to prove everything you claim.

(a) $f : \mathbb{R} \smallsetminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x}{x-1}$.

(b) $g : \mathbb{R} \smallsetminus \{\pm 1\} \to \mathbb{R}$ by $g(x) = f(x^2)$.

(c) $h : A \to \mathbb{R}$ by $h = g|_A$, where $A = [0, 1) \cup (1, \infty)$

(d) $k : \mathbb{R} \to \mathbb{R}$ by $k(x) = x + \lfloor x \rfloor$.

(e) $F : \mathbb{R} \smallsetminus \{\pm 1\} \to \mathbb{R}$ by $F(x) = \dfrac{x}{x^2 - 1}$

Answers/Proofs. (a) **One-to-one**: Given $x, y \in \mathbb{R} \smallsetminus \{1\}$ with $f(x) = f(y)$, we have $x/(x-1) = y/(y-1)$, and so cross-multiplying, we have $xy - x = xy - y$. Adding $x + y - xy$ to both sides gives $x = y$. QED

Not onto: We claim that $1 \in \mathbb{R}$ is not in the range of f. Indeed, given $x \in \mathbb{R} \smallsetminus \{1\}$, if $f(x) = 1$, then $x/(x-1) = 1$. Multiplying by $x - 1$ gives $x = x - 1$, and subtracting x gives $0 = -1$, a contradiction. QED

(b) **Not one-to-one**: $-2 \neq 2$ are both in the domain, and $g(-2) = f(4) = g(2)$. QED

Not onto: Again, 1 is not in the range of g; if x is a point in the domain such that $g(x) = 1$, then $x^2/(x^2 - 1) = 1$, giving $x^2 = x^2 - 1$ and hence $0 = -1$, a contradiction. QED

(c) **One-to-One**: Given $x, y \in A$ with $h(x) = h(y)$, we have $g(x) = g(y)$, and hence $f(x^2) = f(y^2)$. That is, $x^2/(x^2-1) = y^2/(y^2-1)$. Cross-multiplying gives $x^2y^2 - x^2 = x^2y^2 - y^2$, and so subtracting x^2y^2 gives $x^2 = y^2$. Since $x, y \geq 0$, we have $x = y$. QED

Not onto: Yet again, and for the same reason, 1 is not in the range of h. (If $h(x) = 1$ for some $x \in A$, then x is also in the domain of g, and so $g(x) = 1$, which contradicts the above.) QED

(d) Note: it will help to write $x = (x - \lfloor x \rfloor) + \lfloor x \rfloor$, with the observation that $x - \lfloor x \rfloor \in [0, 1)$, and $\lfloor x \rfloor \in \mathbb{Z}$.

One-to-One: Given $x, y \in \mathbb{R}$ such that $k(x) = k(y)$, we have

$$(x - \lfloor x \rfloor) + 2\lfloor x \rfloor = (y - \lfloor y \rfloor) + 2\lfloor y \rfloor, \quad \text{so} \quad 2(\lfloor x \rfloor - \lfloor y \rfloor) = (y - \lfloor y \rfloor) - (x - \lfloor x \rfloor).$$

In the second equation, the left side is an even integer, while the right side is the difference of two elements of $[0, 1)$ and hence lies in $(-1, 1)$. The only even integer in that interval is 0, so both sides are 0. Thus, $\lfloor x \rfloor = \lfloor y \rfloor$, and $(x - \lfloor x \rfloor) = (y - \lfloor y \rfloor)$. Adding, we get $x = y$. QED

Not onto: We claim that 1 is not in the range of k. Given $x \in \mathbb{R}$, let $t = x - \lfloor x \rfloor \in [0, 1)$, and let $n = 2\lfloor x \rfloor$, which is an even integer. Then $k(x) = n + t$. If $n \geq 2$, then $k(x) \geq 2$ is not 1; and if $n \leq 0$, then $k(x) < 1$ is not 1. QED

(e) **Not one-to-one**: The points $-1/2$ and 2 are both in the domain and are different, but

$$F(-1/2) = \frac{-1/2}{(1/4) - 1} = \frac{-1/2}{-3/4} = \frac{2}{3} = \frac{2}{2^2 - 1} = F(2).$$

QED

[Note: there are a lot of ways to do this, none of them obvious, but a lot of them easy to find. I did this by arbitrarily picking $x = 2$ in the domain, and then solving the equation $F(t) = F(2)$ to find the other root.]

Onto: Given $y \in \mathbb{R}$, consider two cases. First, if $y = 0$, then 0 is in the domain, and $F(0) = 0 = y$. Otherwise, if $y \neq 0$, then let $x = (1 + \sqrt{1 + 4y^2})/(2y)$, which is a real number because the denominator $2y$ is nonzero, and $1 + 4y^2 > 0$ has a real square root. In addition, the numerator is

$$1 + \sqrt{1 + 4y^2} > 1 + \sqrt{4y^2} = 1 + |2y| > |2y|.$$

In particular, the numerator is *not* $\pm 2y$, and hence x is not ± 1. That is, x is in the domain of F. Finally, since

$$x^2 - 1 = \frac{1 + 2\sqrt{1 + 4y^2} + 1 + 4y^2}{4y^2} - 1 = \frac{2 + 2\sqrt{1 + 4y^2}}{4y^2} = \frac{x}{y},$$

we have

$$F(x) = \frac{x}{x^2 - 1} = \frac{x}{x/y} = y.$$

QED

[I hope it's clear that I thought of that choice of x by just applying the quadratic formula to the equation $F(x) = y$.]

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13. (a) Find the range R of the function h in #12(c). If we now view the target set of h as being R, find a formula for the inverse of h.

(b) Do the same for the function k in #12(d).

Answers/Proofs. (a): We claim the image of h is $R = (-\infty, 0] \cup (1, \infty)$. To prove $h(A) = R$:

($\subseteq$): Given $y \in h(A)$, write $y = h(x)$ for some $x \in A$. If $x \in [0, 1)$, then $x^2 - 1 < 0$ and $x^2 \geq 0$, so $y = x^2/(x^2 - 1) \leq 0$. That is, $y \in (-\infty, 0] \subseteq R$. Otherwise, we have $x \in (1, \infty)$, so that $x^2 > x^2 - 1 > 0$; thus, $y = x^2/(x^2 - 1) > 1$, so $y \in (1, \infty) \subseteq R$.

($\supseteq$): Given $y \in R$, we claim that $y/(y-1) \geq 0$. After all, $y \neq 1$, so $y/(y-1)$ is a real number. If $y \leq 0$, then $y-1 < 0$ also, and hence $y/(y-1) \geq 0$. Otherwise, since $y \in R$, we have $y > 1$, and hence $y-1 > 0$, so that $y/(y-1) > 0$, proving our claim.
Thus, we can define $x = \sqrt{y/(y-1)} \geq 0$. Note that $y \neq y-1$, and hence $x \neq 1$; so $x \in A$. And

$$h(x) = f(x^2) = f\Big(\frac{y}{y-1}\Big) = \frac{y/(y-1)}{[y/(y-1)]-1} = \frac{y}{y-(y-1)} = y.$$

QED

Finally, the formula for the inverse is $h^{-1}(y) = \sqrt{y/(y-1)}$. The proof of ($\supseteq$) above showed that $h \circ h^{-1} = \mathrm{id}_R$. Meanwhile, given $x \in A$, we have

$$h^{-1} \circ h(x) = h^{-1}\Big(\frac{x^2}{x^2-1}\Big) = \sqrt{\frac{x^2/(x^2-1)}{[x^2/(x^2-1)]-1}} = \sqrt{\frac{x^2}{x^2-(x^2-1)}} = \sqrt{x^2} = |x| = x,$$

where the last equality is because $x \in A$, and hence $x \geq 0$. QED

[FYI: I thought of the set R by looking at where some test points went under h and guessing that intervals mapped to intervals.]

(b): We claim the image of k is $R = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1)$. To prove $k(\mathbb{R}) = R$:

($\subseteq$): Given $y \in k(\mathbb{R})$, write $y = k(x)$ for some $x \in \mathbb{R}$. Let $m = \lfloor x \rfloor \in \mathbb{Z}$ and $t = x - m \in [0,1)$. Then

$$y = k(x) = x + m = t + 2m \in [2m, 2m+1) \in \bigcup_{n\in\mathbb{Z}} [2n, 2n+1).$$

($\supseteq$): Given $y \in R$, there is some $n \in \mathbb{Z}$ such that $y \in [2n, 2n+1)$. Letting $t = y - 2n$, then, we have $t \in [0,1)$. Define $x = n + t \in \mathbb{R}$. Then $\lfloor x \rfloor = n$, and therefore

$$k(x) = x + n = 2n + t = y.$$

QED

Finally, [motivated by the proof of ($\supseteq$) above, with a little simplification I did in my scratchwork], we claim the formula for the inverse of k is

$$k^{-1} : R \to \mathbb{R} \quad \text{by} \quad k^{-1}(y) = y - \frac{1}{2}\lfloor y \rfloor.$$

Given $y \in R$, there is some $n \in \mathbb{Z}$ such that $y \in [2n, 2n+1)$, so that $n = \lfloor y \rfloor$, and $y - 2n \in [0,1)$. In particular,

$$\lfloor y - n \rfloor = \lfloor n + (y - 2n) \rfloor = n.$$

Thus,

$$k \circ k^{-1}(y) = k\Big(y - \frac{1}{2}\lfloor y \rfloor\Big) = k(y-n) = y - n + \lfloor y - n \rfloor = y - n + n = y,$$

proving that $k \circ k^{-1} = \mathrm{id}_R$.
Meanwhile, given $x \in \mathbb{R}$, write $x = n + t$ with $n \in \mathbb{Z}$ and $t \in [0,1)$. Then

$$k^{-1}(2n+t) = 2n + t - \frac{1}{2}\lfloor 2n+t \rfloor = 2n + t - \frac{1}{2}(2n) = 2n + t - n = n + t = x,$$

and therefore

$$k^{-1} \circ k(x) = k^{-1}\big(x + \lfloor x \rfloor\big) = k^{-1}(x+n) = k^{-1}(2n+t) = x,$$

proving that $k^{-1} \circ k = \mathrm{id}_{\mathbb{R}}$. QED

14. Define $f : (-3, 1] \to (5, 23]$ by $f(x) = \dfrac{3x^2+23}{x^2+1}$.

(a) Prove that f is indeed a function from $(-3, 1]$ to $(5, 23]$

(b) Prove that f is onto.

(c) Prove that the inverse image $f^{-1}((5, 13])$ is $(-3, -1] \cup \{1\}$.

Proofs. (a): Given $x \in (-3, 1]$, we have $f(x) \in \mathbb{R}$ since the denominator $x^2 + 1$ is nonzero.
In addition, we have $20x^2 \geq 0$, so that $3x^2 + 23 \leq 23x^2 + 23$, and hence (dividing by $x^2+1 > 0$), we have $f(x) \leq 23$.
Finally, we also have $|x| < 3$, and therefore $x^2 < 9$, so that $2x^2 < 18$ and hence $3x^2+23 > 5x^2+5$. Again dividing by $x^2 + 1 > 0$, it follows that $f(x) > 5$.
Thus, we have shown $f(x) \in (5, 23]$. QED (a)

(b): Given $y \in (5, 23]$, define $t = \dfrac{23-y}{y-3}$, which is in $\mathbb{R}$ since $y \neq 3$ and hence the denominator is nonzero.
Also observe that $23 - y \geq 0$ and that $y - 3 > 2 > 0$, so that $t \geq 0$.
Furthermore, since $y > 5$, we have $10y > 50$, and hence $9y-27 > 23-y$. Dividing by $y-3 > 0$, it follows that $9 > t$. Thus, $t \in [0, 9)$.
Define $x = -\sqrt{t} \in (-3, 0] \subseteq (-3, 1]$. Then

$$f(x) = \frac{3x^2+23}{x^2+1} = \frac{3t+23}{t+1} = \frac{\dfrac{3(23-y)}{y-3} + 23}{\dfrac{23-y}{y-3} + 1} = \frac{3(23-y) + 23(y-3)}{23-y+(y-3)} = \frac{20y}{20} = y$$

QED (b)

(c): ($\subseteq$): Given $x \in$ LHS, we have $f(x) \in (5, 13]$, so in particular, $f(x) \leq 13$.
That is, $\dfrac{3x^2+23}{x^2+1} \leq 13$, and hence (multiplying by $x^2+1 > 0$) we have $3x^2+23 \leq 13x^2+13$. It follows that $10 \leq 10x^2$, and hence $x^2 \geq 1$. That is, $|x| \geq 1$.
If $x > 0$, then we have $x \geq 1$ but also of course x lies in $(-3, 1]$, the domain of f. Thus, $x = 1 \in (-3, -1] \cup \{1\}$.
Otherwise, we have $x \leq 0$, so that $x \leq -1$, and hence $x \in (-3, -1] \subseteq (-3, -1] \cup \{1\}$. QED ($\subseteq$)
($\supseteq$): Given $x \in$ RHS, we have $|x| \geq 1$ and hence $x^2 \geq 1$. Therefore, $10 \leq 10x^2$, and hence $3x^2 + 23 \leq 13x^2 + 13$. Dividing by $x^2 + 1 > 0$, it follows that $f(x) \leq 13$. In addition, we saw in part (a) that $f(x) > 5$, and hence $f(x) \in (5, 13]$. That is, $x \in f^{-1}((5, 13])$. QED ($\supseteq$) QED (c)

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15. Let $f : A \to B$ and $g : B \to C$ be functions, and let $S \subseteq A$ and $T \subseteq C$ be subsets. Prove that

$$(g \circ f)(S) = g\big(f(S)\big) \quad \text{and} \quad (g \circ f)^{-1}(T) = f^{-1}\big(g^{-1}(T)\big).$$

Proof, First Equality. ($\subseteq$) Given $c \in (g \circ f)(S)$, there is some $a \in S$ such that $g \circ f(a) = c$. Thus, $f(a) \in f(S)$, and hence

$$c = g \circ f(a) = g\big(f(a)\big) \in g\big(f(S)\big).$$

($\supseteq$) Given $c \in g(f(S))$, there is some $b \in f(S)$ such that $g(b) = c$. Hence, there is some $a \in S$ such that $f(a) = b$. Thus,

$$c = g(b) = g\big(f(a)\big) = g \circ f(a) \in (g \circ f)(S).$$

Proof, Second Equality. ($\subseteq$) Given $a \in (g \circ f)^{-1}(T)$, we have $g \circ f(a) \in T$ by definition. Let $b = f(a) \in B$. Then

$$g(b) = g\big(f(a)\big) \in T,$$

and hence $b \in g^{-1}(T)$. Since $f(a) = b$, it follows that $a \in f^{-1}(g^{-1}(T))$.
($\supseteq$) Given $a \in f^{-1}(g^{-1}(T))$, let $b = f(a)$, so that $b \in g^{-1}(T)$ by definition. Thus,

$$g \circ f(a) = g\big(f(a)\big) = g(b) \in T,$$

and hence $a \in (g \circ f)^{-1}(T)$. QED

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16. Let $f : A \to A$ and $g : A \to B$ be functions. Assume that g is invertible, and let $h = g \circ f \circ g^{-1} : B \to B$.

(a) Prove that $h \circ h = g \circ f \circ f \circ g^{-1}$.

(b) If f is invertible, prove that h is invertible, and that $h^{-1} = g \circ f^{-1} \circ g^{-1}$.

Proof. (a) Both $h \circ h$ and $g \circ f \circ f \circ g^{-1}$ are clearly functions from B to B. Moreover, for any $b \in B$, we have

$$\begin{aligned} h \circ h(b) &= (g \circ f \circ g^{-1}) \circ (g \circ f \circ g^{-1})(b) = g \circ f \circ (g^{-1} \circ g) \circ f \circ g^{-1}(b) \\ &= g \circ f \circ \mathrm{id}_A \circ f \circ g^{-1}(b) = g \circ f \circ f \circ g^{-1}(b), \end{aligned}$$

where the second equality was by the associativity of $\circ$. QED
(b) By Theorems from class (and from the book), g^{-1} is invertible with inverse g; and for any F, G invertible, $G \circ F$ is invertible with inverse $F^{-1} \circ G^{-1}$. Thus,

$$h^{-1} = (g \circ f \circ g^{-1})^{-1} = (g^{-1})^{-1} \circ f^{-1} \circ g^{-1} = g \circ f^{-1} \circ g^{-1}.$$

QED

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17. Let $f : A \to B$ and $g : B \to C$ be functions. We saw in Theorem 6.2.6 that if f and g are both invertible, then $g \circ f : A \to C$ is invertible. Prove that the converse is false.
That is, give examples of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is invertible but at least one of f or g is not invertible.

(Dis)Proof. Let $A = C = \{1\}$ and $B = \{1, 2\}$. Define $f : A \to B$ by $f(1) = 1$, and define $g : B \to C$ by $g(1) = g(2) = 1$. Then $g \circ f : A \to C$ is given by

$$g \circ f(1) = g\big(f(1)\big) = g(1) = 1.$$

We claim the function $h : C \to A$ given by $h(1) = 1$ is the inverse of $g \circ f$. Certainly the domain and target of h are correct. Moreover, for every $x \in A$, we have $x = 1$, and hence

$$h \circ (g \circ f)(x) = h\big(g \circ f(1)\big) = h(1) = 1 = x.$$

Similarly, for every $x \in C$, we have $x = 1$, and hence

$$(g \circ f) \circ h(x) = (g \circ f)\big(h(1)\big) = g \circ f(1) = 1 = x.$$

Thus, $h = (g \circ f)^{-1}$, proving the claim. In particular, $g \circ f$ is invertible.
However, $f : A \to B$ is not onto, because $2 \in B$ is not in the range $f(A)$. Thus, f is not invertible. QED
[Note: g isn't invertible either (because it's not one-to-one), but no need to say that; we're already done. Also note: there are a lot of other correct counterexamples that could be used to prove the desired converse is false.]

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18. Define $g : \mathbb{R} \smallsetminus \{2\} \to \mathbb{R}$ by $g(x) = \dfrac{4x}{x-2}$. Prove that:

(a) g is not onto, (b) $g\big([-2, 1]\big) = [-4, 2]$, (c) $g\big((2, 6]\big) = [6, \infty)$

Proofs. (a): Pick $y = 4 \in \mathbb{R}$. If there were some $x \in \mathbb{R} \setminus \{2\}$ such that $g(x) = 4$, then $4x/(x-2) = 4$, so that $4x = 4x - 8$, and hence $0 = -8$, a contradiction. Thus, no such x exists, so g is not onto. QED

(b): ($\subseteq$): Given $y \in$ LHS, there is some $x \in [-2, 1]$ such that $y = g(x)$. Since $x \geq -2$, we have $2x \geq -4$ and hence $4x \geq 2x - 4 = 2(x-2)$. Noting that $x \leq 1$, we have $x - 2 < 0$, and therefore $4x/(x-2) \leq 2$; that is $y \leq 2$. Similarly, since $x \leq 1$, we have $8x \leq 8$, so that $4x \leq -4x + 8$. Again because $x - 2 < 0$, we have $y = 4x/(x-2) \geq -4$ Thus, $y \in [-4, 2]$. QED ($\subseteq$)

($\supseteq$): Given $y \in [-4, 2]$, let $x = 2y/(y-4)$. Since $y \geq -4$, we have $2y \geq y - 4$. Dividing by $y - 4 < 0$, we have $x \leq 1$. Meanwhile, since $y \leq 2$, we have $4y \leq 8$ and hence $2y \leq -2y + 8$. Again dividing by $y - 4 < 0$, we have $x \geq -2$. Thus, $x \in [-2, 1]$. Moreover,

$$g(x) = \frac{4x}{x-2} = \frac{4 \cdot 2y/(y-4)}{2y/(y-4) - 2} = \frac{8y}{2y - 2(y-4)} = \frac{8y}{8} = y.$$

Hence, $y = g(x) =\in g([-2, 1])$. QED

($\subseteq$): Given $y \in$ LHS, there is some $x \in (2, 6]$ such that $g(x) = y$. Since $x \leq 6$, we have $2x \leq 12$ and therefore $4x \geq 6x - 12$. Dividing by $x - 2 > 0$, we get $g(x) \geq 6$, i.e., $y \in [6, \infty)$. QED ($\subseteq$)

($\supseteq$): Given $y \in [6, \infty)$, let $x = 2y/(y-4)$. Since $y \geq 6$, we have $24 \leq 4y$, and hence $2y \leq 6y - 24$. Dividing by $y - 4 > 0$, we have $x \leq 6$. Meanwhile, we have $2y > 2y - 8$, and hence, dividing by $y - 4 > 0$, we have $x > 2$. Thus, $x \in (2, 6]$. Moreover,

$$g(x) = \frac{4x}{x-2} = \frac{4 \cdot 2y/(y-4)}{2y/(y-4) - 2} = \frac{8y}{2y - 2(y-4)} = \frac{8y}{8} = y.$$

Hence, $y = g(x) =\in g([6, \infty))$. QED

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19. Define $F : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ by $F(x) = \dfrac{5x - 5}{x + 1}$.
Define $G : [2, 3] \to [3, 4]$ by $G(x) = F(x^2)$. Prove that:

(a) G is indeed a function.

(b) G is bijective.

Proofs. (a): Given $x \in [2, 3]$, we have $x^2 \in [4, 9]$. In particular, $x^2 \neq -1$; thus, $F(x^2)$ is defined. Since $x^2 \leq 9$, we have $5x^2 - 5 \leq 4x^2 + 4$. Dividing by $x^2 + 1 > 0$, we have $G(x) \leq 4$. Meanwhile, since $x^2 \geq 4$, we have $2x^2 \geq 8$, and hence $5x^2 - 5 \geq 3x^2 + 3$. Dividing by $x^2 + 1 > 0$, we have $G(x) \geq 3$. Thus, $G(x)$ is defined and belongs to $[3, 4]$. QED

(b): (one-to-one): Given $s, t \in [2, 3]$ with $G(s) = G(t)$, we have

$$\frac{5s^2 - 5}{s^2 + 1} = \frac{5t^2 - 5}{t^2 + 1}, \quad \text{and so} \quad 5s^2t^2 + 5s^2 - 5t^2 - 5 = 5s^2t^2 + 5t^2 - 5s^2 - 5.$$

Thus, $10s^2 = 10t^2$, so that $s^2 = t^2$. Since $s, t > 0$, we have $s = t$.
(onto): Given $y \in [3, 4]$, let $x = \sqrt{(5+y)/(5-y)}$. Note that x is indeed defined, because $5 - y \geq 5 - 4 > 0$, and $5 + y \geq 5 + 3 > 0$.
Since $y \leq 4$, we have $10y \leq 40$, and thereore $5 + y \leq 45 - 9y$. Dividing by $5 - y > 0$, we have $(5+y)/(5-y) \leq 9$. Taking square roots, we have $x \leq 3$. Similarly, since $y \geq 3$, we have $5y \geq 15$, and thereore $5 + y \geq 20 - 4y$. Dividing by $5 - y > 0$, we have $(5+y)/(5-y) \geq 4$. Taking square roots, we have $x \geq 2$. Thus, $x \in [2, 3]$, and

$$G(x) = F\Big(\frac{5+y}{5-y}\Big) = \frac{5(5+y)/(5-y) - 5}{(5+y)/(5-y) + 1} = \frac{5(5+y) - 5(5-y)}{(5+y) + (5-y)} = \frac{10y}{10} = y \qquad \text{QED}$$

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20. Prove Theorem 6.2.8(a): Let $f : A \to B$ be a function, and let $C, D \subseteq A$ be subsets. If $C \subseteq D$, then (prove that) $f(C) \subseteq f(D)$.

Proof. Given $y \in f(C)$, there exists $x \in C$ such that $f(x) = y$. Then $x \in C \subseteq D$.
Hence, $y = f(x) \in f(D)$. QED

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21. Let $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \dfrac{4x}{x^2+1}$. Prove the following equalities of sets.

(a) $h^{-1}\big([2,6)\big) = \{1\}$ (b) $h\big((-\infty,-1]\big) = [-2,0)$

Proof. (a): ($\subseteq$): Given $x \in h^{-1}([2,6))$, we have $h(x) \in [2,6)$, so in particular $h(x) \geq 2$. That is, $\dfrac{4x}{x^2+1} \geq 2$, so that $4x \geq 2x^2 + 2$, since $x^2 + 1 > 0$. Thus, $2x^2 - 4x + 2 \leq 0$, i.e. $2(x-1)^2 \leq 0$. But since $x \in \mathbb{R}$, we have $2(x-1)^2 \geq 0$, so that $2(x-1)^2 = 0$, and hence $x = 1 \in \{1\}$.

($\supseteq$): Given $x \in \{1\}$, we have $x = 1$, so that $h(x) = \dfrac{4}{1+1} = 2 \in [2,6)$. Therefore, $x \in h^{-1}([2,6))$. QED (a)

(b): ($\subseteq$): Given $y \in h((-\infty,-1])$, there exists $x \in (-\infty,-1]$ such that $y = h(x)$. That is, $x \leq -1$, and $y = \dfrac{4x}{x^2+1}$. Hence, $y < 0$, since $x^2 + 1 > 0$ and $4x \leq -1 < 0$.
We also have $2x^2 + 4x + 2 = 2(x+1)^2 \geq 0$, and therefore $4x \geq -2x^2 - 2 = -2(x^2+1)$ Dividing by $x^2 + 1 > 0$, it follows that $y = h(x) \geq -2$. Thus, $y \in [-2,0)$.

($\supseteq$): Given $y \in [-2,0)$, note that we have $y \neq 0$ and $|y| \leq 2$, so that $4 - y^2 \geq 0$. Thus, we may define $x = \dfrac{2+\sqrt{4-y^2}}{y} \in \mathbb{R}$.

We claim that $x \leq -1$. To see this, first observe that since $y \geq -2$, we have $-2 - y \leq 0 \leq \sqrt{4-y^2}$. Thus, $2 + \sqrt{4-y^2} \geq -y$, so that multiplying both sides by $1/y < 0$ gives $x = \dfrac{2+\sqrt{4-y^2}}{y} \leq -1$, as claimed. That is, $x \in (-\infty,-1]$.

Finally, we have $x^2 + 1 = \dfrac{4 + 4\sqrt{4-y^2} + 4 - y^2}{y^2} + 1 = \dfrac{8 + 4\sqrt{4-y^2}}{y^2} = \dfrac{4x}{y}$. Rearranging, we have $y = \dfrac{4x}{x^2+1} = h(x)$, so that $y \in h((-\infty,-1])$. QED

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22. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be real sequences, and suppose that there is some $m \in \mathbb{N}$ such that $a_m = b_m$ and $a_{m+1} = b_{m+1}$.

(a) If both sequences are arithmetic, prove that $a_n = b_n$ for all $n \in \mathbb{N}$.

(b) If both sequences are geometric, prove that $a_n = b_n$ for all $n \in \mathbb{N}$.

Proof. (a): By hypothesis, there are constants $c, d, s, t \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, we have $a_n = cn + d$ and $b_n = sn + t$. Thus,

$$c = \big(c(m+1) + d\big) - (cm + d) = a_{m+1} - a_m = b_{m+1} - b_m = \big(s(m+1) + t\big) - (sm + t) = s,$$

and therefore also

$$d = (cm + d) - cm = a_m - cm = b_m - sm = (sm + t) - sm = t.$$

Hence, for any $n \in \mathbb{N}$, we have $a_n = cn + d = sn + t = b_n$. QED (a)

(b): By hypothesis, there are constants $c, r, d, s \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, we have $a_n = cr^n$ and $b_n = ds^n$.
If either $c = 0$ or $r = 0$, then $ds^m = b_m = a_m = 0$, so that either $d = 0$ or $s = 0$. Conversely, if either $d = 0$ or $s = 0$, then $cr^m = a_m = b_m = 0$, so that $c = 0$ or $r = 0$. In that case, then $a_n = 0 = b_n$ for all $n \in \mathbb{N}$, and we are done.

Thus, we may assume that $c, d, r, s \neq 0$. We have

$$r = \frac{cr^{m+1}}{cr^m} = \frac{a_{m+1}}{a_m} = \frac{b_{m+1}}{b_m} = \frac{ds^{m+1}}{ds^m} = s,$$

and therefore also

$$c = \frac{cr^m}{r^m} = \frac{a_m}{r^m} = \frac{b_m}{s^m} = \frac{ds^m}{s^m} = d.$$

Hence, for any $n \in \mathbb{N}$, we have $a_n = cr^n = ds^n = b_n$. QED (b)

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23. Let $(a_n)_{n=1}^{\infty}$ be a real sequence. Suppose that for $n \in \mathbb{N}$, we have $|a_n| \leq 1000$.
   (a) If $(a_n)_{n=1}^{\infty}$ is arithmetic, prove that it is a constant sequence.
   (b) Show, by example, that if $(a_n)_{n=1}^{\infty}$ is geometric, it is **not** necessarily constant.

Proof. (a): By hypothesis, there are constants $b, c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $a_n = bn + c$.

We claim that $b = 0$. To prove this, suppose not. Then there is an integer $m \in \mathbb{N}$ such that $m > 1 + 2000/|b|$. Hence,

$$|a_m - a_1| = |(bm + c) - (b + c)| = |b|(m - 1) > |b| \cdot \frac{2000}{|b|} = 2000.$$

However, since $a_1, a_m \in [-1000, 1000]$, we have $|a_m - a_1| \leq 2000$, contradicting the previous line.

Thus, we must have $b = 0$, as claimed. Therefore, for all $n \in \mathbb{N}$, we have $a_n = c$, i.e., the sequence is constant. QED (a)

(b): Let $a_n = (-1)^n$, so that $(a_n)_{n=1}^{\infty}$ is geometric with $a_n = cr^n$ for $c = 1$ and $r = -1$. We have $|a_n| = 1 \leq 1000$ for all $n \in \mathbb{N}$, but $a_1 = -1 \neq 1 = a_2$, so that sequence is not constant.

[**Note**: There are many other examples. For any choice of $r \in [-1, 1)$ and any $c \in \mathbb{R}$ such that $|cr| \leq 1000$, we have $|cr^n| \leq |cr| \leq 1000$ for all $n \in \mathbb{N}$.]

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24. Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing real sequence. Prove that any subsequence of $(a_n)_{n=1}^{\infty}$ is also strictly decreasing.

Proof. Given a subsequence $(a_{n_i})_{i=1}^{\infty}$, we have $n_1 < n_2 < n_3 < \cdots$ by definition. For each $i \geq 1$, we have $a_{n_i} > a_{n_{i+1}}$ because $n_i < n_{i+1}$ and $(a_n)_{n=1}^{\infty}$ is strictly decreasing. Thus, the subsequence $(a_{n_i})_{i=1}^{\infty}$ is strictly decreasing. QED

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25. Let $(a_n)_{n=1}^{\infty}$ be a sequence. For each of the functions $f : \mathbb{N} \to \mathbb{N}$ below, determine whether or not $(b_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, where $b_n = a_{f(n)}$.
   (a) $f(n) = 5^n + n!$ (b) $f(n) = n^2 - 4n + 8$ (c) $f(n) = n^2 - 2n + 7$

Proof. (a): $\boxed{\text{YES, subsequence}}$ as follows.
Given $n \in \mathbb{N}$, we have $(n + 1)! = (n + 1) \cdot n! \geq n!$ and $5^{n+1} > 5^n$. Thus, $f(n + 1) > f(n)$. QED (a)

(b): $\boxed{\text{NO, not subsequence}}$ as follows.
We have $f(1) = 5 \geq 4 = f(2)$, so f is *not* strictly increasing, so (b_n) is *not* a subsequence. QED (b)

(a): $\boxed{\text{YES, subsequence}}$ as follows.
Given $n \in \mathbb{N}$, we have

$$f(n+1) - f(n) = (n+1)^2 - 2(n+1) + 7 - (n^2 - 2n + 7) = n^2 + 2n + 1 - 2n - 2 + 7 - n^2 + 2n - 7 = 2n - 1 > 0.$$

Thus, $f(n + 1) > f(n)$. QED (c)

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26. Let (a_n), (b_n), and (c_n) be sequences of real numbers. Suppose that (a_n) is a subsequence of (b_n), and that (b_n) is a subsequence of (c_n). Prove that (a_n) is a subsequence of (c_n).

Proof. Because (b_n) is a subsequence of (c_n), there is a strictly increasing sequence $(n_j)_{j=1}^{\infty}$ of positive integers such that $b_j = c_{n_j}$ for each $j \in \mathbb{N}$.
Similarly, because (a_n) is a subsequence of (b_n), there is a strictly increasing sequence $(m_i)_{i=1}^{\infty}$ of positive integers such that $a_i = b_{m_i}$ for each $i \in \mathbb{N}$.
For each integer $i \in \mathbb{N}$, define $N_i = n_{m_i} \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, we have $m_{i+1} > m_i$ (since (m_i) is strictly increasing), and hence

$$N_{i+1} = n_{m_{(i+1)}} > n_{m_i} = N_i,$$

since (n_j) is strictly increasing. Thus, $(N_i)_{i=1}^{\infty}$ is a strictly increasing sequence of positive integers. By definition, then, $(c_{N_i}))_{i=1}^{\infty}$ is a subsequence of (c_n). In addition, for each $i \in \mathbb{N}$, we have

$$a_i = b_{m_i} = c_{n_{m_i}} = c_{N_i},$$

so that (a_i) is indeed a subsequence of (c_n). QED

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27. Let (a_n), (b_n), and (c_n) be sequences of real numbers. Suppose that there are integers $M, N \geq 1$ such that

- for all $n \geq M$, we have $a_n \leq b_n$, and
- for all $n \geq N$, we have $b_n \leq c_n$.

Prove that there is an integer $K \geq 1$ such that for all $n \geq K$, we have $a_n \leq c_n$.

Proof. Let $K = \max\{M, N\} \in \mathbb{N}$. Given an integer $n \geq K$, we have $n \geq M$ and $n \geq N$.
Therefore $a_n \leq b_n \leq c_n$. QED

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28. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $(b_n)_{n=1}^{\infty}$ be a strictly increasing geometric sequence of positive integers, so that $(a_{b_n})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

If $(a_n)_{n=1}^{\infty}$ is a strictly increasing geometric sequence, prove that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is definitely **not** geometric.

Proof. Since the two sequences are both geometric, there are real numbers $r, s \in \mathbb{R}$ such that for every $n \geq 2$, we have $a_n = a_1 r^{n-1}$ and $b_n = b_1 s^{n-1}$.
Since $a_2 > a_1$, we have $a_1(r-1) > 0$. In particular, $a_1 \neq 0$ and $r \neq 1$. Since $a_3 > a_2$, we also have $a_1(r^2 - r) > 0$. Dividing by $a_1(r-1) > 0$, it follows that $r > 0$.
By similar reasoning applied to the strictly increasing sequence (b_n), we also have $b_1 \neq 0$ and $s \neq 1$.
Suppose (towards a contradiction) that the subsequence (a_{b_n}) were geometric. Then the ratio of the first two terms $a_{b_1} = a_1 r^{b_1 - 1}$ and $a_{b_2} = a_1 r^{b_2 - 1}$ would equal the ratio of the third term $a_{b_3} = a_1 r^{b_3 - 1}$ and the second term. (None of these terms is 0, since $a_1, r \neq 0$.) That is,

$$r^{b_2 - b_1} = \frac{a_1 r^{b_2 - 1}}{a_1 r^{b_1 - 1}} = \frac{a_1 r^{b_3 - 1}}{a_1 r^{b_2 - 1}} = r^{b_3 - b_2}.$$

Since $r > 0$ with $r \neq 1$, it follows that $b_2 - b_1 = b_3 - b_2$, and hence $b_1(s-1) = b_1(s^2 - s)$. Dividing both sides by $b_1(s-1) \neq 0$, we have $1 = s$, a contradiction. QED

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29. Give an example of a real, non-constant, geometric sequence $(a_n)_{n=1}^{\infty}$, and a strictly increasing geometric sequence of positive integers $(b_n)_{n=1}^{\infty}$ such that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is also geometric.

Solution/Proof. Let $a_n = (-1)^n$ for all $n \in \mathbb{N}$, and let $b_n = 2^n$ for all n. Then both (a_n) and (b_n) are real, geometric sequences (since -1 and 2 are real constants). Moreover, b_n is in fact a sequence of positive integers, since $2^n \in \mathbb{N}$ for all $n \in \mathbb{N}$.

We also have $a_1 = -1 \neq 1 = a_2$, so (a_n) is non-constant. And for every $n \in \mathbb{N}$, we have

$$b_{n+1} = 2^{n+1} = 2 \cdot 2^n > 2^n = b_n,$$

so that (b_n) is strictly increasing. Finally, for all $n \in \mathbb{N}$, we have

$$a_{b_n} = (-1)^{b_n} = (-1)^{2^n} = 1 = 1^n,$$

so that $(a_{b_n}) = (1^n)$ is a (constant) geometric sequence. QED

Note: More generally, we could have picked $a_n = a(-1)^n$ (where $a \in \mathbb{R} \smallsetminus \{0\}$ is a nonzero constant), and $b_n = k^n$ (where $k \geq 2$ is a constant integer). But those are the only examples. In particular, the only way to get this to work is to have the ratio r for (a_n) to be $r = -1$, so that (a_n) is nonconstant but (a_{b_n}) is constant and hence geometric.