Math 220, Section 03, Fall 2025 — Professor Rob Benedetto

Solutions to Midterm Exam 2

1. **(15 points)** Let $m, n \in \mathbb{Z}$ be integers. Suppose that $40|(m-11)$ and that $40|(n+9)$. Prove that $40|(mn+19)$.

Proof. By hypothesis, there exist integers $a, b \in \mathbb{Z}$ such that $m - 11 = 40a$ and $n + 9 = 40b$. Thus,

$$\begin{aligned} mn + 19 &= (40a+11)(40b-9) + 19 = 40^2ab + 11 \cdot 40b - 9 \cdot 40a - 99 + 19 \\ &= 40^2ab + 40(11b - 9a) - 80 = 40(40ab + 11b - 9a - 2). \end{aligned}$$

Since $40ab + 11b - 9a - 2 \in \mathbb{Z}$, we have $40|(mn+19)$. QED

---

2. **(**20 points) Let $m, n \in \mathbb{N}$ be positive integers.
Let $d = \gcd(m, n)$, the greatest common divisor of m and n.
Prove that $m|n$ if and only if $d = m$.

Proof (Method 1). ($\Rightarrow$) By hypothesis, $m|n$. In addition, $m|m$ since $m \cdot 1 = m$. Thus, m is a common divisor of m and n, so $m \leq d$ (since d is the *greatest* common divisor).
On the other hand, since $d|m$, we have $m \geq d$. Thus, $m = d$ QED ($\Rightarrow$)

($\Leftarrow$) Since $d|n$ and $m = d$, we have $m|n$ QED ($\Leftarrow$) QED

Proof (Method 2). Let $p_1, \ldots, p_k$ be all the primes that divide either m or n, so that, taking the prime factorizations of m and n, we have

$$m = p_1^{r_1} \cdots p_k^{r_k} \quad \text{and} \quad n = p_1^{s_1} \cdots p_k^{s_k},$$

where $r_1, \ldots, r_k \geq 0$ and $s_1, \ldots, s_k \geq 0$ are nonnegative integers.
By a theorem from the book, we have $d = p_1^{t_1} \cdots p_k^{t_k}$, where $t_i = \min\{r_i, s_i\}$ for each i.

($\Rightarrow$) Since $m|n$, we have $r_i \leq s_i$ for each i. Therefore, for each i, we have $t_i = \min r_i, s_i = r_i$, and hence $d = p_1^{t_1} \cdots p_k^{t_k} = p_1^{r_1} \cdots p_k^{r_k} = m$ QED ($\Rightarrow$)

($\Leftarrow$) By the uniqueness of prime factorization, the fact that $d = m$ implies that for each i, we have $r_i = t_i = \min\{r_i, s_i\} \leq s_i$. Hence, $m = p_1^{r_1} \cdots p_k^{r_k} \big| p_1^{s_1} \cdots p_k^{s_k} = n$ QED ($\Leftarrow$) QED

---

3. **(23 points)** Let $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{3\}$ by $f(x) = \dfrac{3x+5}{x+1}$.
You may take my word for it that f **is** actually a function.
Prove that f is onto.

Proof. Given $y \in \mathbb{R} \setminus \{3\}$, let $x = \dfrac{5-y}{y-3}$, which is in $\mathbb{R}$ since $y - 3 \neq 0$.
If $x = -1$, then $5 - y = -1(y-3)$, i.e., $5 - y = -y + 3$, so $5 = 3$, a contradiction.
Thus, we must have $x \neq -1$, so $x \in \mathbb{R} \setminus \{-1\}$. We compute:

$$f(x) = \frac{3\left(\dfrac{5-y}{y-3}\right) + 5}{\dfrac{5-y}{y-3} + 1} = \frac{3(5-y) + 5(y-3)}{(5-y) + (y-3)} = \frac{15 - 3y + 5y - 15}{2} = \frac{2y}{2} = y$$ QED

Note: Of course, I came up with the formula for x in my scratchwork: starting from $y = f(x)$ and solving for x.

4. **(20 points)** Let $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^2 - 6$.
Prove that $h^{-1}([-10, 10]) = [-4, 4]$.

Proof. ($\subseteq$): Given $x \in$ LHS, then $-10 \le x^2 - 6 \le 10$. The second inequality gives $x^2 \le 16$, so that $|x| \le 4$, and hence $-4 \le x \le 4$. Thus, $x \in [-4, 4]$. QED ($\subseteq$)

($\supseteq$): Given $x \in [-4, 4]$, then $|x| \le 4$, so $x^2 \le 16$. Since we also have $x^2 \ge 0$, it follows that $-10 \le -6 \le x^2 - 6 = h(x) \le 10$. Thus, $h(x) \in [-10, 10]$, and hence $x \in$ LHS. QED ($\supseteq$) QED

---

5. **(22 points)** Let $(a_n)_{n=1}^{\infty}$ be a sequence that is increasing.
Suppose that $(a_n)_{n=1}^{\infty}$ has a subsequence that is constant.
Prove that $(a_n)_{n=1}^{\infty}$ is *eventually constant*. That is, prove that there is some integer $M \in \mathbb{N}$ such that for all $n \ge M$, we have $a_n = a_M$.

Proof. By hypothesis, there is a function $f : \mathbb{N} \to \mathbb{N}$ so that $f(n) > f(m)$ whenever $n > m$, and such that $(a_{f(n)})_{n=1}^{\infty}$ is constant. This last condition means that $a_{f(n)} = a_{f(1)}$ for all $n \in \mathbb{N}$. In addition, since f is increasing, we have $f(n) \ge n$ for each $n \in \mathbb{N}$.

Let $M = f(1) \in \mathbb{N}$. Then given any integer $n \ge M$, we have

$$a_M \le a_n \le a_{f(n)} = a_{f(1)} = a_M,$$

where the first inequality is because (a_n) is an increasing sequence, and the second is because f is increasing and (a_n) is increasing. Thus, $a_n = a_M$. QED

OPTIONAL BONUS. (2 points.) Let $S = \{(a_n)_{n=1}^{\infty} \mid \forall n \in \mathbb{N}, a_n \in \mathbb{R}\}$ be the set of all real sequences. Define $f : S \to S$ by

$$f\big((a_n)_{n=1}^{\infty}\big) = (b_n)_{n=1}^{\infty}, \qquad \text{where} \quad b_n = a_1 + a_2 + \cdots + a_n.$$

Prove that f is an invertible function by finding a formula for $f^{-1} : S \to S$ and proving that it is indeed the inverse of f.

Proof. Define $g : S \to S$ by

$$g\big((a_n)_{n=1}^{\infty}\big) = (c_n)_{n=1}^{\infty}, \qquad \text{where} \quad c_n = \begin{cases} a_1 & \text{if } n = 1, \\ a_n - a_{n-1} & \text{if } n \ge 2. \end{cases}$$

Since each c_n is a real number, we do indeed have $g((a_n)) \in S$ for each $(a_n) \in S$, so g is indeed a function from S to S. We will now show that g is the inverse of f.

Given $(a_n) \in S$, let $(b_n) = f((a_n))$, so that $b_n = a_1 + a_2 + \cdots + a_n$ for each $n \in \mathbb{N}$. Define $(c_n) = g(f((a_n))) = g((b_n))$. Then by definition of g, we have $c_1 = b_1 = a_1$, and for $n \ge 2$,

$$c_n = b_n - b_{n-1} = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = a_n.$$

Thus, we have shown that $c_n = a_n$ for all $n \in \mathbb{N}$, and hence $(c_n) = (a_n)$ as sequences. That is, $g(f((a_n)) = (a_n)$.

Conversely, given $(a_n) \in S$, let $(c_n) = g((a_n))$, so that $c_1 = a_1$ and $c_n = a_n - a_{n-1}$ for $n \ge 2$. Define $(d_n) = f(g((a_n))) = f((c_n))$. That is, for each $n \in \mathbb{N}$, we have $d_n = c_1 + \cdots + c_n$. We claim that for each $n \in \mathbb{N}$, we have $d_n = a_n$. It will then follow that $f(g((a_n))) = (a_n)$, at which point we will be done. So it remains only to prove our claim, which we now do by induction on $n \ge 1$.

Base Case: We have $d_1 = c_1 = a_1$, so the claim is true for $n = 1$.

Inductive Step: Suppose we know the claim for some particular $n \in \mathbb{N}$; we wish to prove it for $n+1$. Since $n + 1 \geq 2$, we have

$$d_{n+1} = c_1 + \cdots + c_n + c_{n+1} = d_n + c_{n+1} = a_n + (a_{n+1} - a_n) = a_{n+1},$$

where the first equality is by definition of d_{n+1}, the second is by definition of d_n, and the third is by the inductive hypothesis together with the fact that $c_{n+1} = a_{n+1} - a_n$. QED