Math 220, Section 03, Fall 2025 Professor Rob Benedetto

Practice Problems for Midterm Exam 2
(A little more difficult, and much longer, than the real exam)

1. Prove or disprove the following statement: for all integers $m, n \in \mathbb{Z}$, if $m|n^2$, then $m|n$.

2. Suppose that $a, b, c \in \mathbb{Z}$ are nonzero integers such that $143a + 217b = c$. Prove that $\gcd(a, b)|c$.

3. Prove that $4|(13^n - 1)$ for every $n \in \mathbb{N}$.

4. Prove that $7|(10^n - 3^n)$ for every $n \in \mathbb{N}$.

5. Let $n \in \mathbb{N}$ be a positive integer, and write its prime factorization as $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, and $r_1, r_2, \ldots, r_k \in \mathbb{N}$. Prove that n is a perfect cube if and only if each of the integers $r_1, r_2, \ldots, r_k$ is divisible by 3.

6. Let $a, b, c \in \mathbb{N}$ be positive integers. Suppose that $c = a^2$, but also $c = b^3$. Prove that there is some $n \in \mathbb{N}$ such that $c = n^6$.

7. Let $p, q \in \mathbb{N}$ be prime numbers. Suppose that $p|q$. Prove that $p = q$.

8. Let $m, n \in \mathbb{Z}$ be positive integers. Suppose that $20|(m - 7)$ and $20|(n - 11)$. Prove that $20|(mn - 17)$.

9. Let $a, b, c \in \mathbb{Z}$, and suppose that $a|(15b + 31c)$ and that $a|15$. Prove that $a|c$.

10. Let $a, b, m, n \in \mathbb{N}$, and suppose that $am = bn$ and that $\gcd(a, b) = 1$. Prove that there is some $k \in \mathbb{N}$ such that $n = ka$ and $m = kb$.

11. Determine whether each of the following supposed functions is actually a function.

(a) $f : (0, \infty) \to (0, \infty)$ by $f(x) = \dfrac{x - 1}{\lceil x \rceil}$ [$\lceil x \rceil$ denotes the ceiling function of x]

(b) $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by $g(A) = \{n + 3 \,|\, n \in A\}$.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ by $h(A) = \min A$. [$\min A$ denotes the smallest element of A.]

(d) $k : \mathcal{P}(\mathbb{N}) \smallsetminus \{\varnothing\} \to \mathbb{N}$ by $k(A) = \max A$. [$\max A$ denotes the largest element of A.]

(e) $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ by $F(n) = \{m \in \mathbb{N} \,|\, m \text{ is a divisor of } n\}$.

12. For each of the following functions, decide whether or not it is injective, and also whether or not it is surjective. If it is both, find a formula for its inverse function. Don't forget to prove everything you claim.

(a) $f : \mathbb{R} \smallsetminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x}{x - 1}$.

(b) $g : \mathbb{R} \smallsetminus \{\pm 1\} \to \mathbb{R}$ by $g(x) = f(x^2)$.

(c) $h : A \to \mathbb{R}$ by $h = g|_A$, where $A = [0, 1) \cup (1, \infty)$

(d) $k : \mathbb{R} \to \mathbb{R}$ by $k(x) = x + \lfloor x \rfloor$.

(e) $F : \mathbb{R} \smallsetminus \{\pm 1\} \to \mathbb{R}$ by $F(x) = \dfrac{x}{x^2 - 1}$

13. (a) Find the range R of the function h in #12(c). If we now view the target set of h as being R, find a formula for the inverse of h.

(b) Do the same for the function k in #12(d).

14. Define $f : (-3, 1] \to (5, 23]$ by $f(x) = \dfrac{3x^2 + 23}{x^2 + 1}$.

(a) Prove that f is indeed a function from $(-3, 1]$ to $(5, 23]$

(b) Prove that f is onto.

(c) Prove that the inverse image $f^{-1}((5, 13])$ is $(-3, -1] \cup \{1\}$.

15. Let $f : A \to B$ and $g : B \to C$ be functions, and let $S \subseteq A$ and $T \subseteq C$ be subsets. Prove that

$$(g \circ f)(S) = g\big(f(S)\big) \quad \text{and} \quad (g \circ f)^{-1}(T) = f^{-1}\big(g^{-1}(T)\big).$$

16. Let $f : A \to A$ and $g : A \to B$ be functions. Assume that g is invertible, and let $h = g \circ f \circ g^{-1} : B \to B$.

(a) Prove that $h \circ h = g \circ f \circ f \circ g^{-1}$.

(b) If f is invertible, prove that h is invertible, and that $h^{-1} = g \circ f^{-1} \circ g^{-1}$.

17. Let $f : A \to B$ and $g : B \to C$ be functions. We saw in Theorem 6.2.6 that if f and g are both invertible, then $g \circ f : A \to C$ is invertible. Prove that the converse is false. That is, give examples of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is invertible but at least one of f or g is not invertible.

18. Define $g : \mathbb{R} \smallsetminus \{2\} \to \mathbb{R}$ by $g(x) = \dfrac{4x}{x - 2}$. Prove that:

(a) g is not onto.

(b) $g\big([-2, 1]\big) = [-4, 2]$

(c) $g\big((2, 6]\big) = [6, \infty)$

19. Define $F : \mathbb{R} \smallsetminus \{-1\} \to \mathbb{R}$ by $F(x) = \dfrac{5x - 5}{x + 1}$.
Define $G : [2, 3] \to [3, 4]$ by $G(x) = F(x^2)$. Prove that:

(a) G is indeed a function.

(b) G is bijective.

20. Prove Theorem 6.2.8(a): Let $f : A \to B$ be a function, and let $C, D \subseteq A$ be subsets. If $C \subseteq D$, then (prove that) $f(C) \subseteq f(D)$.

21. Let $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \dfrac{4x}{x^2 + 1}$. Prove the following equalities of sets.

(a) $h^{-1}\big([2, 6)\big) = \{1\}$ (b) $h\big((-\infty, -1]\big) = [-2, 0)$

22. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be real sequences, and suppose that there is some $m \in \mathbb{N}$ such that $a_m = b_m$ and $a_{m+1} = b_{m+1}$.

(a) If both sequences are arithmetic, prove that $a_n = b_n$ for all $n \in \mathbb{N}$.

(b) If both sequences are geometric, prove that $a_n = b_n$ for all $n \in \mathbb{N}$.

23. Let $(a_n)_{n=1}^{\infty}$ be a real sequence. Suppose that for $n \in \mathbb{N}$, we have $|a_n| \leq 1000$.

(a) If $(a_n)_{n=1}^{\infty}$ is arithmetic, prove that it is a constant sequence.

(b) Show, by example, that if $(a_n)_{n=1}^{\infty}$ is geometric, it is **not** necessarily constant.

24. Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing real sequence. Prove that any subsequence of $(a_n)_{n=1}^{\infty}$ is also strictly decreasing.

25. Let $(a_n)_{n=1}^{\infty}$ be a sequence. For each of the functions $f : \mathbb{N} \to \mathbb{N}$ below, determine whether or not $(b_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, where $b_n = a_{f(n)}$.

(a) $f(n) = 5^n + n!$ (b) $f(n) = n^2 - 4n + 8$ (c) $f(n) = n^2 - 2n + 7$

26. Let (a_n), (b_n), and (c_n) be sequences of real numbers. Suppose that (a_n) is a subsequence of (b_n), and that (b_n) is a subsequence of (c_n). Prove that (a_n) is a subsequence of (c_n).

27. Let (a_n), (b_n), and (c_n) be sequences of real numbers. Suppose that there are integers $M, N \geq 1$ such that

- for all $n \geq M$, we have $a_n \leq b_n$, and
- for all $n \geq N$, we have $b_n \leq c_n$.

Prove that there is an integer $K \geq 1$ such that for all $n \geq K$, we have $a_n \leq c_n$.

28. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $(b_n)_{n=1}^{\infty}$ be a strictly increasing geometric sequence of positive integers, so that $(a_{b_n})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

If $(a_n)_{n=1}^{\infty}$ is a strictly increasing geometric sequence, prove that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is definitely **not** geometric.

29. Give an example of a real, non-constant, geometric sequence $(a_n)_{n=1}^{\infty}$, and a strictly increasing geometric sequence of positive integers $(b_n)_{n=1}^{\infty}$ such that the subsequence $(a_{b_n})_{n=1}^{\infty}$ is also geometric.

[**Note**: As always, don't forget to prove all of your claims.]