Math 220, Section 01, Fall 2026 Professor Rob Benedetto

Solutions to Midterm Exam 2

1. **(18 points)** Prove that for every $n \in \mathbb{N}$, we have $8|(19^n - 3^n)$.

Proof, by induction on n.
Base Case: For $n = 1$, we have $19^n - 3^n = 19 - 3 = 16 = 8 \cdot 2$, which is divisible by 8.
Inductive Step: Given that it's true for $n \geq 1$, we may write $19^n - 3^n = 8m$ for some $m \in \mathbb{Z}$, and hence

$$19^{n+1} - 3^{n+1} = 19(19^n - 3^n) + 19 \cdot 3^n - 3^{n+1} = 19(8m) + (19-3) \cdot 3^n = 8(19m) + 8(2 \cdot 3^n) = 8(19m + 2 \cdot 3^n).$$

Since $19m + 2 \cdot 3^n \in \mathbb{Z}$, we have $8|(19^{n+1} - 3^{n+1})$. QED

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2. **(24 points)** Define $f : (2, \infty) \to (1, \infty)$ by $f(x) = \dfrac{x}{x-2}$.
You may take my word for it that f actually **is** a function.
Prove that f is one-to-one and onto.

Proof. **(1-1):** Given $x_1, x_2 \in (2, \infty)$ such that $f(x_1) = f(x_2)$, then $\dfrac{x_1}{x_1 - 2} = \dfrac{x_2}{x_2 - 2}$.
Cross-multiplying, we obtain $x_1 x_2 - 2x_1 = x_1 x_2 - 2x_2$, and hence $-2x_1 = -2x_2$.
Therefore $x_1 = x_2$. QED 1-1

(onto): Given $y \in (1, \infty)$, let $x = \dfrac{2y}{y-1}$, which **is** a real number since $y \neq 1$.
Since $0 > -2$, we have $2y > 2y - 2 = 2(y-1)$. Because $y > 1$, we have $y - 1 > 0$.
Therefore, dividing by $y - 1$ gives $x = \dfrac{2y}{y-1} > 2$. That is, $x \in (2, \infty)$.
Finally, we observe that

$$f(x) = \frac{\dfrac{2y}{y-1}}{\dfrac{2y}{y-1} - 2} = \frac{2y}{2y - (2y-2)} = \frac{2y}{2} = y$$ QED

Note: Of course, the scratchwork for the onto portion (to decide how to choose x) was:

$$\frac{x}{x-2} = y \;\Leftrightarrow\; x = xy - 2y \;\Leftrightarrow\; x - xy = -2y \;\Leftrightarrow\; x(1-y) = -2y \;\Leftrightarrow\; x = \frac{2y}{y-1}$$

and then that the desired inequality $x > 2$ becomes $\dfrac{2y}{y-1} > 2$, i.e., $2y > 2y - 2$, and then $0 > -2$.

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3. **(18 points)** Let $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^2 + 6$. Prove that $h^{-1}\big([7, 10)\big) = (-2, -1] \cup [1, 2)$.

Proof. ($\subseteq$): Given $x \in h^{-1}\big([7, 10)\big)$, we have $h(x) \in [7, 10)$.
That is, $7 \leq x^2 + 6 < 10$, and hence $1 \leq x^2 < 4$.
Taking square roots, we have $1 \leq |x| < 2$, and hence either $-2 < x \leq -1$ or $1 \leq x < 2$.
That is, $x \in$ RHS.

($\supseteq$): Given $x \in$ RHS, we have $1 \leq |x| < 2$, and hence $1 \leq x^2 < 4$.
Therefore $7 \leq x^2 + 6 < 10$, i.e., $h(x) \in [7, 10)$, and hence $x \in h^{-1}\big([7, 10)\big)$. QED

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4. **(18 points)** Let $n \in \mathbb{N}$ be a positive integer, and write its prime factorization as

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

Suppose that n is a perfect cube, i.e., suppose that there is a positive integer $m \in \mathbb{N}$ such that $n = m^3$. Prove that each of the integers $r_1, r_2, \ldots, r_k$ is divisible by 3.

Proof. Given that n is a perfect cube, there exists $m \in \mathbb{N}$ such that $m^3 = n$. Write the prime factorization of m as $m = q_1^{s_1} \cdots q_\ell^{s_\ell}$, where $q_1, \ldots, q_\ell$ are distinct primes, and each $s_i \in \mathbb{N}$. Then

$$p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = n = m^3 = q_1^{3s_1} \cdots q_\ell^{3s_\ell}.$$

Thus, possibly after re-ordering, the q_i's are precisely the p_i's, and $r_i = 3s_i$ for each i. Thus, each r_i is divisible by 3. QED

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5. **(22 points)** Prove, from the ε-N definition, that $\lim_{n\to\infty} \dfrac{1-6n^2}{2n^2+7} = -3$

Proof. Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $N > 11/\varepsilon$. Given $n \geq N$, we have $N \leq n \leq n^2$, so

$$\left|\frac{1-6n^2}{2n^2+7} - (-3)\right| = \left|\frac{1-6n^2+3(2n^2+7)}{2n^2+7}\right| = \left|\frac{1-6n^2+6n^2+21}{2n^2+7}\right|$$
$$= \left|\frac{22}{2n^2+7}\right| = \frac{22}{2n^2+7} < \frac{22}{2n^2} = \frac{11}{n^2} \leq \frac{11}{N} < \frac{11}{11/\varepsilon} = \varepsilon$$ QED

Note: Of course, the scratchwork here is:

$$\left|\frac{1-6n^2}{2n^2+7} - (-3)\right| = \left|\frac{1-6n^2+6n^2+21}{2n^2+7}\right| = \frac{22}{2n^2+7} < \frac{22}{2n^2} = \frac{11}{n^2},$$

and then noting that to get that $< \varepsilon$, we only need $\dfrac{11}{N} < \varepsilon$, i.e., $N > \dfrac{11}{\varepsilon}$

OPTIONAL BONUS. (2 points) Define $f : \mathbb{N} \to \mathcal{P}(\mathcal{P}(\mathbb{N}))$ by

$$f(n) = \big\{A \in \mathcal{P}(\mathbb{N}) \,\big|\, n \in A\big\}.$$

Find a function $g : \mathcal{P}(\mathcal{P}(\mathbb{N})) \to \mathbb{N}$ such that $g \circ f = \mathrm{id}_{\mathbb{N}}$. (And prove your claims, of course.)

Answer/Proof. Define $G : \mathcal{P}\big(\mathcal{P}(\mathbb{N})\big) \to \mathcal{P}(\mathbb{N})$ and $H : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ by

$$G(\mathcal{C}) = \begin{cases} \bigcap_{A\in\mathcal{C}} A & \text{if } \mathcal{C} \neq \varnothing, \\ \mathbb{N} & \text{if } \mathcal{C} = \varnothing, \end{cases} \quad \text{and} \quad H(B) = \begin{cases} \min B & \text{if } B \neq \varnothing, \\ 1 & \text{if } B = \varnothing. \end{cases}$$

Note that G is certainly a (defined) function. For any $\mathcal{C} \in \mathcal{P}(\mathcal{P}(\mathbb{N}))$, either $\mathcal{C} = \varnothing$ or else it makes sense to take the intersection of the elements of $\mathcal{C}$; either way, $G(\mathcal{C})$ is indeed a subset of $\mathbb{N}$ and hence an element of $\mathcal{P}(\mathbb{N})$.

Also, H is a (defined) function: by the well-ordering principle, every non-empty subset of $\mathbb{N}$ has a least element, and this is what H outputs.

Define $g = H \circ G : \mathcal{P}\big(\mathcal{P}(\mathbb{N})\big) \to \mathbb{N}$.

Given $n \in \mathbb{N}$, we need to show that $H(G(f(n))) = n$.

To prove this, first note that $\{n\} \in f(n)$, since $\{n\} \in \mathcal{P}(\mathbb{N})$ is a subset of $\mathbb{N}$ with $n \in \{n\}$. In particular, $f(n) \neq \varnothing$, and hence $G(f(n)) \subseteq \{n\}$, since $G(f(n))$ is the intersection of all elements of $f(n)$, including $\{n\}$. On the other hand, $\{n\} \subseteq G(f(n))$, since every $A \in f(n)$ has $\{n\} \subseteq A$, and hence the intersection of all such A still contains $\{n\}$. Thus, $G(f(n)) = \{n\}$. As a result,

$$g \circ f(n) = H(G(f(n))) = H(\{n\}) = \min\{n\} = n,$$

as desired. QED