**Solutions to Practice Problems 1**

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 5, 1, 7\}$, $C = \{3, 4, 4, 4, 7\}$. Compute each of the following sets:

**Answers.** Before starting, it will help to rewrite $B = \{1, 2, 5, 7\}$ and $C = \{3, 4, 7\}$. So:
a. $A \cap B = \{1, 2, 5\}$.
b. $A \cap C = \{3, 4\}$.
c. $B \cap C = \{7\}$.
d. $A \cap B \cap C = \varnothing$.
e. $A \cup B = \{1, 2, 3, 4, 5, 7\}$.
f. $(A \cup C) \cap B = \{1, 2, 3, 4, 5, 7\} \cap \{1, 2, 5, 7\} = \{1, 2, 5, 7\}$.
g. $A \smallsetminus C = \{1, 2, 5\}$.
h. $[(B \cup C) \cap A] \cup (C \smallsetminus B) = [\{1, 2, 3, 4, 5, 7\} \cap A] \cup \{3, 4\} = \{1, 2, 3, 4, 5\} \cup \{3, 4\} = \{1, 2, 3, 4, 5\}$.
i. $C \times C = \{(3, 3), (3, 4), (3, 7), (4, 3), (4, 4), (4, 7), (7, 3), (7, 4), (7, 7)\}$.
j. $(A \cap C) \times (B \cap C) = \{3, 4\} \times \{7\} = \{(3, 7), (4, 7)\}$.

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2. Let $A, B, C$ be sets. If $A \subseteq B$, prove that $(C \smallsetminus B) \subseteq (C \smallsetminus A)$.

**Proof**. Given $x \in C \smallsetminus B$. Then $x \in C$, but $x \notin B$.
If we had $x \in A$, then $x \in B$, a contradiction. Thus, $x \notin A$.
Since $x \in C$ and $x \notin A$, we have $x \in C \smallsetminus A$. QED

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3. Show that the converse of problem 2 fails; that is, give examples of set $A, B, C$ such that $(C \smallsetminus B) \subseteq (C \smallsetminus A)$, but $A \not\subseteq B$.

**Answer/Proof**. Choose $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. Then

$$(C \smallsetminus B) = \{3\} \subseteq \{3\} = (C \smallsetminus A),$$

but $A \not\subseteq B$, since $1 \in A$ but $1 \notin B$. QED

[There are *many* ways to do this; just choose your sets so that $A$ is not a subset of $C$.]

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4. For each of the following sets, write out all the elements in the form $\{\text{blah}, \text{blah}, \dots, \text{blah}\}$.

**Answers**.
a. $\varnothing = \{\}$.
b. $\mathcal{P}(\varnothing) = \{\varnothing\}$.
c. $\mathcal{P}\big(\mathcal{P}(\varnothing)\big) = \big\{\varnothing, \{\varnothing\}\big\}$.

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5. Let $A, B, C, D$ be sets. Prove that $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.

**Proof**. ($\subseteq$): Given $(x, y) \in (A \times C) \cap (B \times D)$, we have $(x, y) \in A \times C$, and hence $x \in A$ and $y \in C$; we also have $(x, y) \in B \times D$, and hence $x \in B$ and $y \in D$.
Thus, $x \in A \cap B$, and $y \in C \cap D$. That is, $(x, y) \in (A \cap B) \times (C \cap D)$. QED ($\subseteq$)
($\supseteq$): Given $(x, y) \in (A \cap B) \times (C \cap D)$, we have $x \in A \cap B$ and $y \in C \cap D$.

Thus, $x \in A$ and $y \in C$, so that $(x, y) \in A \times C$. Similarly, $x \in B$ and $y \in D$, so that $(x, y) \in B \times D$. That is, $(x, y) \in (A \times C) \cap (B \times D)$. QED ($\supseteq$)

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6. Let $A, B, C, D$ be sets. Prove that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$.

**Proof**. Given $(x, y) \in (A \times C) \cup (B \times D)$, we have two cases.

**Case 1:** $(x, y) \in A \times C$. Then $x \in A \subseteq A \cup B$, and $y \in C \subseteq C \cup D$.
Hence, $(x, y) \in (A \cup B) \times (C \cup D)$.

**Case 2:** $(x, y) \in B \times D$. Then $x \in B \subseteq A \cup B$, and $y \in D \subseteq C \cup D$.
Hence, $(x, y) \in (A \cup B) \times (C \cup D)$. QED

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7. Give examples of sets $A, B, C, D$ such that $(A \times C) \cup (B \times D) \neq (A \cup B) \times (C \cup D)$.

**Answer/Proof**. Choose $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then

$$(A \times C) \cup (B \times D) = \{(1, 3)\} \cup \{(2, 4)\} = \{(1, 3), (2, 4)\},$$

but

$$(A \cup B) \times (C \cup D) = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

The two sets are not equal because, for example, $(1, 4)$ belongs to the second one but not the first. QED

[Again, there are many ways to do this.]

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8. Find $\bigcup_{t \in (0,\infty)} (-t, \infty)$ and prove your claim.

**Solution/Proof** We claim that $\bigcup_{t \in (0,\infty)} (-t, \infty) = \mathbb{R}$.

**Proof of claim**: ($\subseteq$): Given $x \in$ LHS, there is some $t \in (0, \infty)$ such that $x \in (-t, \infty) \subseteq \mathbb{R}$, and hence $x \in \mathbb{R}$.

($\supseteq$): Given $x \in \mathbb{R}$, we consider two cases.
**Case 1.** If $x \geq 0$, then $x \in (-1, \infty) \subseteq$ LHS
**Case 2.** If $x \not\geq 0$, then $x < 0$. Let $t = 1 - x \in (0, \infty)$. Then $x = 1 - t > -t$. Thus, $x \in (-t, \infty) \subseteq$ LHS. QED (claim)

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9. Find $\bigcup_{t \in (0,\infty)} (t, \infty)$ and prove your claim.

**Solution/Proof**. We claim that $\bigcup_{t \in (0,\infty)} (t, \infty) = (0, \infty)$.

**Proof of claim**: ($\subseteq$): Given $x \in$ LHS, there is some $t \in (0, \infty)$ such that $x \in (t, \infty)$. Thus, $x > t > 0$, and hence $x \in (0, \infty)$.
($\supseteq$): Given $x \in (0, \infty)$, let $t = x/2 \in (0, \infty)$. Then $x > t$, and hence $x \in (t, \infty) \subseteq$ LHS. QED

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10. Prove that $\bigcap_{t \in (0,\infty)} (t, \infty) = \varnothing$.

**Proof**. ($\supseteq$): This is vacuous, since $\varnothing$ has no elements.

($\subseteq$). We proceed by contradiction. Suppose there were an element $x \in$ LHS. Then in particular, $x \in (1, \infty)$, so $x > 0$. Choosing $t = x \in (0, \infty)$, then we have $x \notin (t, \infty)$, and hence $x \notin$ LHS, a contradiction. Thus, the set on the LHS must be empty. QED

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11. For each positive integer $k \in \mathbb{N}$, define

$$S_k = \{n \in \mathbb{N} \,|\, k \text{ divides } n\}.$$

For any integers $k, m \in \mathbb{N}$, prove that

$$k|m \iff S_m \subseteq S_k.$$

**Proof**. ($\Longrightarrow$): Given $k, m \in \mathbb{N}$ such that $k|m$, and given $n \in S_m$, we have $m|n$. That is, there are integers $i, j \in \mathbb{N}$ such that $m = ki$ and $n = mj$. Thus, $n = (ki)j = k(ij)$. Since $ij \in \mathbb{N}$, we have $k|n$, and hence $n \in S_k$.

($\Longleftarrow$): Given $k, m \in \mathbb{N}$ such that $S_m \subseteq S_k$, we have $m = m \cdot 1$, and hence $m|m$. Thus, $m \in S_m \subseteq S_k$, meaning that $k$ divides $m$, i.e., $k|m$. QED

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12. Demonstrate, using a truth table, that $(P \wedge (\sim Q)) \vee ((\sim P) \vee Q)$ is a tautology, i.e., that it is always true, regardless of whether $P$ and $Q$ are true or false.

**Answer/Proof**:

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $P \wedge (\sim Q)$ | $(\sim P) \vee Q$ | $(P \wedge (\sim Q)) \vee ((\sim P) \vee Q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | T | T |
| T | F | F | T | T | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

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13. Prove the following statement: $\forall a \in \mathbb{N}, \exists b, c \in \mathbb{N}$ s.t. $ab = c^3$.

**Proof**. Given $a \in \mathbb{N}$, let $b = a^2 \in \mathbb{N}$, and let $c = a \in \mathbb{N}$. Then $ab = a \cdot a^2 = a^3 = c^3$. QED

[There are many other ways to do this, but this is probably the easiest.]

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14. Give the negation of the following statement: $\forall a \in \mathbb{N}, \exists b, c \in \mathbb{N}$ s.t. $a = bc^3$ and $c \neq 1$.

**Answer**. $\exists a \in \mathbb{N}$ s.t. $\forall b, c \in \mathbb{N}$, either $a \neq bc^3$ or $c = 1$.

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15. Prove the statement you gave as the answer to problem 14 above.

**Proof**. Let $a = 1 \in \mathbb{N}$. Given $b, c \in \mathbb{N}$, if $c = 1$, we are done. Thus, we may assume that $c \geq 2$. Since $b \geq 1$, we have $bc^3 \geq 1 \cdot 2^3 = 8 > 1 = a$, and hence $a \neq bc^3$. QED

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16. Consider the statement "Everybody loves a lover." Let's assume that this statement is true, where a "lover" means anybody who loves somebody. Under that assumption, together with the assumption that Fred loves Jane, prove that everybody loves everybody.

**Proof**. Let $S$ be the set of all people. Given $x, y \in S$, then because Fred is a lover, $y$ must love Fred. Thus, $y$ is a lover, and therefore $x$ loves $y$. QED

17. Find the contrapositive, the converse, and the negation of the following implication statement:

$$\big((S \subseteq T) \vee (a = b)\big) \Rightarrow (a, b) \in S \times T.$$

**Answer**. Contrapositive: $(a, b) \notin S \times T \Rightarrow \big((S \not\subseteq T) \wedge (a \neq b)\big)$.
Converse: $(a, b) \in S \times T \Rightarrow \big((S \subseteq T) \vee (a = b)\big)$.
Negation: $\big((S \subseteq T) \vee (a = b)\big) \wedge \big((a, b) \notin S \times T\big)$.

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18. Let $n \in \mathbb{Z}$. Prove the following are equivalent:

a. $5|(n + 2)$ b. $5|(2n - 1)$ c. $5|(6n + 7)$

**Proof**. (a $\Rightarrow$ b): There is some $m \in \mathbb{Z}$ such that $n + 2 = 5m$.
Thus, $2n - 1 = 2(n + 2) - 4 - 1 = 2(5m) - 5 = 5(2m - 1)$.
Since $2m - 1 \in \mathbb{Z}$, we have $5|(2n - 1)$.

(b $\Rightarrow$ c): There is some $m \in \mathbb{Z}$ such that $2n - 1 = 5m$.
Thus, $6n + 7 = 3(2n - 1) + 10 = 3(5m) + 10 = 5(3m + 2)$.
Since $3m + 2 \in \mathbb{Z}$, we have $5|(6n + 7)$.

(c $\Rightarrow$ a): There is some $m \in \mathbb{Z}$ such that $6n + 7 = 5m$.
Thus, $n + 2 = 6n + 7 - 5n - 5 = 5m - 5n - 5 = 5(m - n - 1)$.
Since $m - n - 1 \in \mathbb{Z}$, we have $5|(n + 2)$. QED

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19. Prove that there is no way to choose real numbers $a, b, c \in \mathbb{R}$ so that the polynomial $f(x) = ax^2 + bx + c$ satisfies $f(0) = 3$, $f(1) = f(-1) = 2$, and $f(2) = 5$.

**Proof**. We proceed by contradiction.
Suppose there were such a polynomial $f(x) = ax^2 + bx + c$. Then $3 = f(0) = c$. In addition $2 = f(1) = a+b+c = a+b+3$, so that $a+b = -1$. Moreover, $2 = f(-1) = a-b+c = a-b+3$, so that $a - b = -1$. Adding these two equations, we have $2a = -2$, and hence $a = -1$; then $b = 0$. Thus, $5 = f(2) = 4a + 2b + c = -4 + 0 + 3 = -1$, a contradiction. QED

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20. For any $x \in \mathbb{R}$, prove that there exists a unique $y \in \mathbb{R}$ such that $5x + 3y = 7$.

**Proof**. (Existence): Given $x \in \mathbb{R}$, let $y = (7 - 5x)/3 \in \mathbb{R}$. Then

$$5x + 3y = 5x + (7 - 5x) = 7.$$

(Uniqueness): Given $x \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$ such that $5x + 3y_1 = 7 = 5x + 3y_2$. Then $3y_1 = 3y_2$, and hence $y_1 = y_2$. QED

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21. Prove, for every $n \in \mathbb{N}$, that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

**Proof.** By induction on $n$.
Base Case: For $n = 1$, LHS $= 1 = 1^2$, as desired.
Inductive Step: Given that the claim holds for $n = k \in \mathbb{N}$, we have

$$1 + 3 + \cdots + \big(2(k + 1) - 1\big) = 1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2,$$

where the second equality is by the inductive hypothesis. QED

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22. Define a sequence $a_1, a_2, a_3, \ldots$ of real numbers by

$$a_1 = 1, \quad \text{and} \quad \text{for } n \geq 1,\ a_{n+1} = 3 - \frac{1}{a_n}.$$

Prove that for every $n \in \mathbb{N} \setminus \{1, 2\}$, we have $a_n \in (2, 3)$, and $a_n > a_{n-1}$.

**Proof**. By induction on $n \geq 3$.

Base Case: We compute $a_2 = 3 - 1/1 = 2$, and $a_3 = 3 - 1/2 = 5/2$. Since $5/2 \in (2, 3)$, and $5/2 > 2 = a_2$, the base case holds.

Inductive Step: Given that the claim holds for some $n = k \geq 3$, we have

$$a_{k+1} = 3 - \frac{1}{a_k} < 3, \quad \text{since } a_k > 2 > 0, \quad \text{and}$$

$$a_{k+1} = 3 - \frac{1}{a_k} > 3 - \frac{1}{a_{k-1}} = a_k > 2.$$

Thus, we have both $a_{k+1} > a_k$, and $a_{k+1} \in (a_k, 3) \subseteq (2, 3)$. QED

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23. Prove, for all $n \in \mathbb{N}$, that the integer $1 + 3^{2n-1}$ is divisible by 4.

**Proof**. By induction on $n$.

Base Case: For $n = 1$, we have $1 + 3^{2-1} = 1 + 3 = 4$, which is divisible by 4.

Inductive Step: Given that the claim holds for $n = k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $1 + 3^{2n-1} = 4m$. Therefore,

$$1 + 3^{2(k+1)-1} = 1 + 3^{2k-1} - 3^{2k-1} + 3^{2k+1} = 4m + 3^{2k-1}(9 - 1) = 4(m + 2 \cdot 3^{2k-1}).$$

Since $m + 2 \cdot 3^{2k-1} \in \mathbb{N}$, the integer $1 + 3^{2(k+1)-1}$ is divisible by 4. QED