Math 220, Section 01, Fall 2026 Professor Rob Benedetto

Solutions to Midterm Exam 1

1. **(18 points, 4 parts)** Let $A = \{1, 3, 6, 7\}$, $B = \{3, 4, 5\}$, and $C = \{1, 4, 8\}$. Compute each of the following sets.

1a. $(A \smallsetminus B) \cap C$ 1b. $A \smallsetminus (B \cap C)$

1c. $(A \cap C) \times B$ 1d. $A \cap (C \times B)$

Solutions. (a): $(A \smallsetminus B) \cap C = \{1, 6, 7\} \cap \{1, 4, 8\} = \boxed{\{1\}}$

(b): $A \smallsetminus (B \cap C) = \{1, 3, 6, 7\} \smallsetminus \{4\} = \boxed{\{1, 3, 6, 7\}}$

$(A \cap C) \times B = \{1\} \times \{3, 4, 5\} = \boxed{\{(1, 3), (1, 4), (1, 5)\}}$

$A \cap (C \times B) = \{1, 3, 6, 7\} \cap (\text{a set of ordered pairs}) = \boxed{\varnothing}$

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2. **(11 points)** Let A_1, A_2, B_1, B_2 be sets. Suppose that $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Prove that

$$A_2 \times (B_2 \smallsetminus B_1) \subseteq (A_2 \times B_2) \smallsetminus (A_1 \times B_1).$$

Proof. Given $(x, y) \in$ LHS, then $x \in A_2$ and $y \in B_2 \smallsetminus B_1$. Then $y \in B_2$, and hence $(x, y) \in A_2 \times B_2$. If $(x, y) \in A_1 \times B_1$, then $y \in B_1$, which is a contradiction since $y \notin B_1$. Therefore $(x, y) \notin A_1 \times B_1$. Thus, we have $(x, y) \in (A_2 \times B_2) \smallsetminus (A_1 \times B_1)$. QED

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3. **(14 points, 2 parts)** Recall that $\mathcal{P}(A)$ denotes the power set of a set A.

3a. (7 points) List all the elements of $\{S \in \mathcal{P}(\{2, 6, 7\}) \,|\, \{2, 6\} \cap S \neq \varnothing\}$.

3b. (7 points) List all the elements of $\{S \in \mathcal{P}(\{2, 6, 7\}) \,|\, \{2, 6\} \subseteq S\}$.

Solutions. (a): We have $\mathcal{P}(\{2, 6, 7\}) = \big\{\varnothing, \{2\}, \{6\}, \{7\}, \{2, 6\}, \{2, 7\}, \{6, 7\}, \{2, 6, 7\}\big\}$. Therefore, $\{S \in \mathcal{P}(\{2, 6, 7\}) \,|\, \{2, 6\} \cap S \neq \varnothing\} = \boxed{\big\{\{2\}, \{6\}, \{2, 6\}, \{2, 7\}, \{6, 7\}, \{2, 6, 7\}\big\}}$

(b): Similarly, we have $\{S \in \mathcal{P}(\{2, 6, 7\}) \,|\, \{2, 6\} \subseteq S\} = \boxed{\big\{\{2, 6\}, \{2, 6, 7\}\big\}}$

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4. **(15 points)** Define a sequence $a_1, a_2, a_3, \ldots$ of real numbers by

$$a_1 = 1, \quad \text{and} \quad \text{for } n \geq 1,\ a_{n+1} = 4a_n + 9.$$

Prove that for every $n \in \mathbb{N}$, we have $a_n = 4^n - 3$.

Proof. By induction on $n \geq 1$.

Base Case. For $n = 1$, we have $a_1 = 1 = 4^1 - 3$.

Inductive Step. Assume the statement is true for some particular $n \geq 1$. Then

$$a_{n+1} = 4a_n + 9 = 4(4^n - 3) + 9 = 4^{n+1} - 12 + 9 = 4^{n+1} - 3.$$ QED

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5. **(20 points, 3 parts)**

5a. (5 points) Give the contrapositive of the following statement, which is about

a real number b and a subset S of $\mathbb{R}$:

$$\big(\exists x \in \mathbb{Z} \text{ s.t. } x^2 + 7x = b\big) \Rightarrow \big((b \in S) \wedge (b - 4 \geq 0)\big).$$

5b. (5 points) Give the negation of the statement:

$$\forall a \in \mathbb{Z}, \exists b, c \in \mathbb{Z} \text{ s.t. } ab^2 \neq c^3 \text{ and } c = 0.$$

5c. (10 points) Prove the statement you gave as your answer to 5b above.

Solutions. (a): The contrapositive is: $\boxed{\big((b \notin S) \vee (b - 4 < 0)\big) \Rightarrow \big(\forall x \in \mathbb{Z},\, x^2 + 7x \neq b\big)}$

(b): The negation is: $\boxed{\exists a \in \mathbb{Z} \text{ s.t. } \forall b, c \in \mathbb{Z}, \text{ we have } ab^2 = c^3 \text{ or } c \neq 0}$

(c): **Proof**. Let $a = 0 \in \mathbb{Z}$.
Given arbitrary $b, c \in \mathbb{Z}$, we consider two cases.

Case 1: $c = 0$. Then $ab^2 = 0 \cdot b^2 = 0 = c^3$.

Case 2: $c \neq 0$. Then we are already done. QED

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6. **(22 points)** Prove that $\displaystyle\bigcup_{n \in \mathbb{N}} \left[4, 9 - \frac{2}{n}\right] = [4, 9)$

Proof. ($\subseteq$): Given $x \in$ LHS. Then there exists $n \in \mathbb{N}$ such that $4 \leq x \leq 9 - \dfrac{2}{n}$. Hence,

$$4 \leq x \leq 9 - \frac{2}{n} < 9,$$

so $x \in [4, 9)$. QED ($\subseteq$)

($\supseteq$): Given $x \in [4, 9)$, then $x < 9$, so $9 - x > 0$.

Pick $n \geq \mathbb{N}$ such that $n > \dfrac{2}{9 - x}$.

Then (since $9 - x > 0$ and $n > 0$), we have $9 - x > \dfrac{2}{n}$, and hence $x < 9 - \dfrac{2}{n}$.

We also have $x \geq 4$, and hence $x \in \left[4, 9 - \dfrac{2}{n}\right] \subseteq$ LHS. QED

OPTIONAL BONUS. (2 points) Prove that $\displaystyle\bigcap_{t \in [0,2)} (5 - 2t, 10 - t) = (5, 8]$

Proof. ($\subseteq$): Given $x \in$ LHS, then (with $t = 0$), we have $x \in (5, 10)$, and hence $x > 5$.

It remains to show that $x \leq 8$. Suppose, towards contradiction, that $x > 8$.

Then let $t = 10 - x$. Observe that

- $t = 10 - x \geq 0$, since $x < 10$, and
- $t = 10 - x < 2$, since $x > 8$.

Thus, $t \in [0, 2)$. However, $x = 10 - t \notin (5 - 2t, 10 - t)$, and therefore $x \notin$ LHS, a contradiction.
Hence, we must have $x \leq 8$, and hence $x \in (5, 8]$. QED ($\subseteq$)

($\supseteq$): Given $x \in (5, 8]$, and given $t \in [0, 2)$, we have

$$5 - 2t \leq 5 < x \leq 8 < 10 - t, \qquad \text{so} \quad x \in (5 - 2t, 10 - t),$$

where the first inequality is because $t \geq 0$, and the fourth is because $t < 2$.
Since $x \in (5 - 2t, 10 - t)$ for all $t \in [0, 2)$, we have $x \in$ LHS. QED