Math 220, Section 03, Fall 2025 — Professor Rob Benedetto

Solutions to Midterm Exam 1

1. **(20 points, 4 parts)** Let $A = \{3, 4, 6\}$, $B = \{1, 3\}$, and $C = \{1, 6\}$. Compute each of the following sets.
Briefly justify. (E.g., if I asked for $(A \cap B) \cup C$, first say what $A \cap B$ is.)

1a. $(A \cup B) \smallsetminus C$ — 1b. $A \cup (B \smallsetminus C)$

1c. $(C \smallsetminus B) \times A$ — 1d. $C \smallsetminus (B \times A)$

Solutions. **(a)**: $A \cup B = \{1, 3, 4, 6\}$, so $(A \cup B) \smallsetminus C = \boxed{\{3, 4\}}$

(b): $B \smallsetminus C = \{3\}$, so $A \cup (B \smallsetminus C) = \boxed{\{3, 4, 6\}}$

(c): $C \smallsetminus B = \{6\}$, so $(C \smallsetminus B) \times A = \boxed{\{(6, 3), (6, 4), (6, 6)\}}$

(d): $B \times A$ is a set of ordered pairs, but C has no ordered pairs, so $C \smallsetminus (B \times A) = C = \boxed{\{1, 6\}}$

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2. **(10 points, 2 parts)** Let S, T, V, W be sets. Suppose that $S \subseteq T$ and $V \subseteq W$.

2a. (5 points) Prove that $S \cap V \subseteq T \cap W$.

2b. (5 points) Prove that $S \cup V \subseteq T \cup W$.

Proofs. **(a)**: Given $x \in S \cap V$, we have $x \in S$, so $x \in T$.
We also have $x \in V$, so $x \in W$.
Thus, $x \in T \cap W$. QED

(b): Given $x \in S \cup V$, we have either $x \in S$ or $x \in V$.
If $x \in S$, then $x \in T \subseteq T \cup W$.
Otherwise, we have $x \in V$, so that $x \in W \subseteq T \cup W$. QED

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3. **(18 points)** Use mathematical induction to prove that for every $n \in \mathbb{N}$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof. By induction on $n \geq 1$.

Base: For $n = 1$, we have LHS $= \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{n}{n+1} =$ RHS QED Base

Ind. Step: Assume it's true for some $n = k \geq 1$. Then

$$\frac{1}{1 \cdot 2} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{1}{1 \cdot 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

(by the inductive hypothesis), so continuing, this is:

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$ QED

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4. **(12 points, 2 parts)** Recall that $\mathcal{P}(S)$ denotes the power set of a set S.

4a. (5 points) List all the elements of $\big\{A \in \mathcal{P}(\{2, 3, 4\}) \,\big|\, 3 \in A\big\}$

4b. (7 points) List all the elements of $\big\{A \in \mathcal{P}(\{7, 8, 9\}) \,\big|\, \{5, 9\} \cup A = \{5, 7, 9\}\big\}$

Solutions. **(a)**: The elements are the subsets of $\{2,3,4\}$ that contain 3, i.e.: $\boxed{\{3\},\{2,3\},\{3,4\},\{2,3,4\}}$

(b): The elements are the subsets of $\{7,8,9\}$ that contain 7 but *not* 8. That is: $\boxed{\{7\},\{7,9\}}$

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5. **(20 points, 3 parts)**
   5a. (5 points) Give the contrapositive of the following statement, which is about a real number c and a subset S of $\mathbb{R}$:

$$\big(\exists x \in \mathbb{Z} \text{ s.t. } x^6 + 3x = c\big) \Rightarrow \big((c \in S) \wedge (c + 2 \geq 0)\big).$$

   5b. (5 points) Give the negation of the following statement:

$$\forall a \in \mathbb{Z}, \exists b, c \in \mathbb{Z} \text{ s.t. } ab \geq c^2 \text{ and } b > 0.$$

   5c. (10 points) Prove the statement you gave as your answer to 5b above.

Solutions/Proof. **(a)**: The contrapositive is

$$\boxed{\big((c \notin S) \vee (c + 2 < 0)\big) \Rightarrow \big(\forall x \in \mathbb{Z},\ x^6 + 3x \neq c\big)}$$

(b): The negation is

$$\boxed{\exists a \in \mathbb{Z} \text{ s.t. } \forall b, c \in \mathbb{Z}, \text{ either } ab < c^2 \text{ or } b \leq 0}$$

(c): **Proof**. Let $a = -1 \in \mathbb{Z}$.
Given $b, c \in \mathbb{Z}$, consider two cases:

Case 1: $b \leq 0$. Then we are already done.

Case 2: Otherwise, we have $b > 0$. Then $ab = -b < 0 \leq c^2$ QED

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6. **(20 points)** Recall the standard notation for intervals: for $a, b \in \mathbb{R}$ with $a < b$,

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad (a,b) = \{x \in \mathbb{R} \mid a < x < b\}, \quad (a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

Prove that $\displaystyle\bigcup_{t \in (0,3)} [5+t, 10] = (5, 10]$.

Proof. ($\subseteq$): Given $x \in$ LHS, there exists $t \in (0,3)$ such that $x \in [5+t, 10]$. Thus,

$$5 < 5 + t \leq x \leq 10,$$

and hence $x \in (5, 10]$. QED ($\subseteq$)

($\supseteq$): Given $x \in (5, 10]$, consider two cases.

Case 1. $x < 8$, so that $5 < x < 8$.
Define $t = x - 5$, so that $0 < t < 3$ and hence $t \in (0,3)$.
We also have $5 + t = x < 8 \leq 10$, so that $x \in [5+t, 10]$. Hence, $x \in$ LHS.

Case 2. Otherwise, we have $x \geq 8$, so that $8 \leq x \leq 10$. Let $t = 2 \in (0,3)$. Then $x \in [7, 10] = [5+t, 10]$, and hence $x \in$ LHS. QED ($\supseteq$)

QED

Alternative proof of ($\supseteq$):

Given $x \in (5,10]$, let $t = \frac{1}{2}(x-5)$

Since $x > 5$, we have $t > 0$. In addition, since $x \leq 10$, we also have $t \leq \frac{1}{2}(10-5) = \frac{5}{2} < 3$. Thus, $t \in (0,3)$.

Furthermore,

$$5 + t = 5 + \frac{1}{2}(x-5) = \frac{1}{2}(x+5) < \frac{1}{2}(x+x) = x \leq 10,$$

where the two inequalities are because $5 < x \leq 10$, since $x \in (5,10]$.

That is, we have $x \in (5+t, 10]$ with $t \in (0,3)$. Hence, $x \in$ LHS. QED ($\supseteq$)

OPTIONAL BONUS. (2 points.) With the same standard interval notation as in Problem 6, prove that

$$\bigcap_{t \in (0,1)} [3+t, 7+2t] = [4,7]$$

Proof. ($\subseteq$): Given $x \in [4,7]$, then for any $t \in (0,1)$, we have

$$3 + t < 4 \leq x \leq 7 < 7 + 2t,$$

so $x \in (3+t, 7+2t)$. This holds for all $t \in (0,1)$, so $x \in$ LHS.

($\supseteq$): Given $x \in$ LHS, then (using $t = \frac{1}{2} \in (0,1)$), we have $x \in [\frac{7}{2}, 8]$, so that $\frac{7}{2} \leq x \leq 8$.

Claim 1: $x \geq 4$

Proof of Claim 1: Suppose (towards contradiction) that $x < 4$.

In that case, choose $t = \frac{1}{2}(x+4) - 3 = \frac{x}{2} - 1$.

Then $t \geq \frac{7}{4} - 1 = \frac{3}{4} > 0$ because $x \geq \frac{7}{2}$. In addition, since $x < 4$, we also have $t < \frac{4}{2} - 1 = 1$. Hence, $t \in (0,1)$. However,

$$3 + t = \frac{x}{2} + 2 = \frac{x+4}{2} > \frac{x+x}{2} = x,$$

so that $x \notin [3+t, 7+2t]$, and hence $x \notin$ LHS. Contradiction! QED Claim 1

Claim 2: $x \leq 7$

Proof of Claim 2: Suppose (towards contradiction) that $x > 7$.

In that case, choose $t = \frac{1}{4}(x-7)$.

Then $t \leq \frac{8-7}{4} = \frac{1}{4} > 0$ because $x \leq 8$. In addition, since $x > 7$, we also have $t = \frac{x-7}{4} > 0$. Hence, $t \in (0,1)$. However,

$$7 + 2t = 7 + \frac{x-7}{2} = \frac{x+7}{2} < \frac{x+x}{2} = x,$$

so that $x \notin [3+t, 7+2t]$, and hence $x \notin$ LHS. Contradiction! QED Claim 2

Combining Claims 1 and 2 gives $4 \leq x \leq 7$, and hence $x \in [4,7]$ QED ($\supseteq$)

QED