Math 220, Section 03, Fall 2025 Professor Rob Benedetto

Practice Problems for Midterm Exam 1

(A little more difficult, and much longer, than the real exam)

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 5, 1, 7\}$, $C = \{3, 4, 4, 4, 7\}$. Compute each of the following sets:

a. $A \cap B$ b. $A \cap C$ c. $B \cap C$
d. $A \cap B \cap C$ e. $A \cup B$ f. $(A \cup C) \cap B$
g. $A \smallsetminus C$ h. $[(B \cup C) \cap A] \cup (C \smallsetminus B)$
i. $C \times C$ j. $(A \cap C) \times (B \cap C)$

2. Let A, B, C be sets. If $A \subseteq B$, prove that $(C \smallsetminus B) \subseteq (C \smallsetminus A)$.

3. Show that the converse of problem 2 fails; that is, give examples of set A, B, C such that $(C \smallsetminus B) \subseteq (C \smallsetminus A)$, but $A \not\subseteq B$.

[Suggestion: use small finite sets.]

4. Write each of the following sets in roster form. That is, for each of the following sets, write out all the elements in the form $\{\text{blah}, \text{blah}, \ldots, \text{blah}\}$. Here, $\mathcal{P}(T)$ means the power set of the set T.

a. $\varnothing$ b. $\mathcal{P}(\varnothing)$ c. $\mathcal{P}\big(\mathcal{P}(\varnothing)\big)$

5. Let A, B, C, D be sets. Prove that $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.

6. Let A, B, C, D be sets. Prove that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$.

7. Give examples of sets A, B, C, D such that $(A \times C) \cup (B \times D) \neq (A \cup B) \times (C \cup D)$.

8. Find $\displaystyle\bigcup_{t \in (0,\infty)} (-t, \infty)$ and prove your claim.

9. Find $\displaystyle\bigcup_{t \in (0,\infty)} (t, \infty)$ and prove your claim.

10. Prove that $\displaystyle\bigcap_{t \in (0,\infty)} (t, \infty) = \varnothing$.

[*Suggestion*: For $(\subseteq)$, use proof by contradiction. That is, suppose there is some $x \in$ LHS and produce a contradiction.]

11. For each positive integer $k \in \mathbb{N}$, define

$$S_k = \{n \in \mathbb{N} \mid k \text{ divides } n\}.$$

For any integers $k, m \in \mathbb{N}$, prove that

$$k|m \iff S_m \subseteq S_k.$$

[Recall that for integers $k, m \in \mathbb{Z}$, the notation $k|m$ means that k divides m, i.e., that there exists an integer $n \in \mathbb{Z}$ such that $m = kn$.]

12. Demonstrate, using a truth table, that $(P \wedge (\sim Q)) \vee ((\sim P) \vee Q)$ is a tautology, i.e., that it is always true, regardless of whether P and Q are true or false.

13. Prove the following statement: $\forall a \in \mathbb{N}, \exists b, c \in \mathbb{N}$ s.t. $ab = c^3$.

14. State (in symbolic mathematical notation) the **negation** of the following statement:

$$\forall a \in \mathbb{N}, \exists b, c \in \mathbb{N} \text{ s.t. } a = bc^3 \text{ and } c \neq 1.$$

15. Prove the statement you gave as the answer to problem 14 above.

16. Consider the statement "Everybody loves a lover." Let's assume that this statement is true, where a "lover" means anybody who loves somebody. Under that assumption, together with the assumption that Fred loves Jane, prove that everybody loves everybody.

17. Find the contrapositive, the converse, and the negation of the following implication statement:

$$\big((S \subseteq T) \vee (a = b)\big) \Rightarrow (a, b) \in S \times T.$$

18. Let $n \in \mathbb{Z}$. Prove the following are equivalent:

a. $5|(n+2)$ b. $5|(2n-1)$ c. $5|(6n+7)$

[Recall the definition of the notation $k|m$ for "k divides m," which also appears in problem 11 above.]

19. Prove that there is no way to choose real numbers $a, b, c \in \mathbb{R}$ so that the polynomial $f(x) = ax^2 + bx + c$ satisfies $f(0) = 3$, $f(1) = f(-1) = 2$, and $f(2) = 5$.

20. For any $x \in \mathbb{R}$, prove that there exists a unique $y \in \mathbb{R}$ such that $5x + 3y = 7$.

21. Prove, for every $n \in \mathbb{N}$, that $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

22. Define a sequence $a_1, a_2, a_3, \ldots$ of real numbers by

$$a_1 = 1, \quad \text{and} \quad \text{for each } n \geq 1,\ a_{n+1} = 3 - \frac{1}{a_n}.$$

Prove that for every $n \in \mathbb{N} \setminus \{1, 2\}$, we have $a_n \in (2, 3)$, and $a_n > a_{n-1}$.

23. Prove, for all $n \in \mathbb{N}$, that the integer $1 + 3^{2n-1}$ is divisible by 4.