Math 211, Sections 01,05 Fall 2018

Solutions to Practice Problems 3

1. For the following integrals, sketch the region of integration, and then reverse the order of integration.

**Answers.** (a). The region for \( \int_0^1 \int_{1-x}^{1-x^2} f(x, y) \, dy \, dx \) is bounded by \( y = 1-x^2 \) above and by \( y = 1-x \) below. (The two curves intersect at \((0,1)\) and \((1,0)\); see me for a sketch.) Solving for \( x \), those two curves can be written as \( x = 1-y \) (to the left) and \( x = \sqrt{1-y} \) (to the right). So the new integral is \( \int_0^1 \int_{\sqrt{1-y}}^{1} f(x, y) \, dx \, dy \).

(b). The region for \( \int_0^{\ln 3} \int_{e^y}^{3} g(x, y) \, dx \, dy \) is bounded by \( x = e^y \) to the left and above, by \( x = 3 \) to the right, and by \( y = 0 \) below. The curve \( x = e^y \) is of course \( y = \ln x \), which intersects \( y = 0 \) and \((1,0)\), and \( x = 3 \) at \((3,\ln 3)\). So the new integral is \( \int_1^{\ln x} \int_0^{3 \cos \theta} g(x, y) \, dx \, dy \).

2. Compute \( \iint_{D} 2x + y \, dA \), where \( D \) is the region in the plane bounded by the line \( y = 3 - 2x \) and the hyperbola \( xy = 1 \).

**Answer.** Rewriting the hyperbola as \( y = 1/x \), we see that the two curves intersect when \( 1/x = 3-2x \), i.e., \( 2x^2 - 3x + 1 = 0 \), i.e., \((2x-1)(x-1) = 0\), i.e., \( x = 1/2 \) and \( x = 1 \). A sketch shows the line is on top, and the hyperbola is below. So the integral is \( \int_{1/2}^{1} \int_{1/x}^{3-2x} 2x + y \, dy \, dx = \int_{1/2}^{1} 2xy + \frac{1}{2}y^2 \bigg|_{y=1/x}^{3-2x} \, dx = \int_{1/2}^{1} 6x - 4x^2 + \frac{1}{2}(3-2x)^2 - 2 - \frac{1}{2x^2} \, dx \)

\[
= \int_{1/2}^{1} -2x^2 + \frac{5}{2} - \frac{1}{2x^2} \, dx = -2 \left( \frac{3}{3} \right) + \frac{5}{2} + \frac{1}{2} - \frac{1}{12} - \frac{5}{4} - 1 \left( = \frac{1}{6} \right)
\]

3. Compute \( \iint_{D} x^2 y \, dA \), where \( D \) is the region in the first quadrant inside the circle \( x^2 + y^2 = 4 \) and outside the circle \( x^2 + y^2 = 1 \).

**Answer.** In polar coordinates, the outer circle is \( r = 2 \), and the inner circle is \( r^2 = r \cos \theta \), or equivalently, \( r = \cos \theta \). The two curves never intersect (since \( \cos \theta < 2 \) always); since we are restricted to the first quadrant, then, \( \theta \) runs from \( 0 \) to \( \pi/2 \). Thus, the integral is

\[
\int_0^{\pi/2} \int_0^{2\cos \theta} (r \cos \theta)^2 (r \sin \theta) \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{2\cos \theta} r^4 \cos^2 \theta \sin \theta \, dr \, d\theta
\]

\[
= \int_0^{\pi/2} \int_0^{2\cos \theta} r^5 \cos^2 \theta \sin \theta \bigg|_{r=0}^{\cos \theta} \, d\theta = \int_0^{\pi/2} (32 \cos^2 \theta - \cos^7 \theta) \sin \theta \, d\theta
\]

\[
[u = \cos \theta, du = -\sin \theta \, d\theta] \quad \frac{1}{2} \int_0^{1} (32u^2 - u^7) \, du = -\frac{1}{5} \left[ \frac{32}{3} u^3 - \frac{1}{8} u^8 \right]_0^1 = \frac{1}{5} \left( \frac{32}{3} - \frac{1}{8} \right).
\]

[On an exam, you can leave a messy sum of fractions like that. (And I'll make sure that it doesn't get that bad anyhow.) But for the record, the above mess simplifies to 253/120.]

4. Compute \( \iint_{D} 4 + x^2 \, dA \), where \( D \) is the region in the plane bounded by \( y = x^2 + 1 \) and \( y = 3 - x^2 \).

**Answer.** The two curves intersect when \( x^2 + 1 = 3 - x^2 \), i.e., \( 2x^2 = 2 \), i.e., \( x^2 = 1 \), i.e., \( x = \pm 1 \). A sketch shows \( y = 3 - x^2 \) is on top, and \( y = x^2 + 1 \) is below. So the integral is

\[
\int_{-1}^{1} \int_{x^2+1}^{3-x^2} 4 + x^2 \, dy \, dx = \int_{-1}^{1} (4 + x^2)y \bigg|_{y=x^2+1}^{3-x^2} \, dx = \int_{-1}^{1} (4 + x^2)(2 - 2x^2) \, dx
\]

\[
= \int_{-1}^{1} 8 - 6x^2 - 2x^4 \, dx = 8x - 2x^3 - \frac{2}{5}x^5 \bigg|_{-1}^{1} = 8 - 2 - \frac{2}{5} + 8 - 2 - \frac{2}{5} \left( = \frac{56}{5} \right).
\]
5. Compute \( \int \int_D x^3 + xy^2 \, dA \), where \( D \) is the region in the first quadrant bounded by the circle \( x^2 + y^2 = 1 \) and the lines \( y = x \) and \( y = \sqrt{3}x \).

**Answer.** In polar coordinates, the line \( y = x \) in the first quadrant is \( \theta = \pi/4 \), and (since \( \tan \pi/3 = \sqrt{3} \)), the line \( y = \sqrt{3}x \) in the first quadrant is \( \theta = \pi/3 \). Meanwhile, \( x^3 + xy^2 = x(x^2 + y^2) = r^3 \cos \theta \).

Thus, the integral is

\[
\int_{\pi/4}^{\pi/3} \int_0^1 r^4 \cos \theta \, dr \, d\theta = \left( \sin \frac{\pi}{4} \right) \left( \frac{r^5}{5} \right) \bigg|_0^1 = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \left( \frac{1}{5} - 0 \right) = \frac{3 - \sqrt{2}}{10}.
\]

6. Compute \( \int \int_D x^2 + y \, dA \), where \( D \) is the region in the plane bounded above by the curves \( y = x^2 \) and \( y = 6 - x \) and below by the \( x \)-axis.

**Answer.** [If we try to integrate \( dy \, dx \), the upper limit of integration on the inner integral would be a piecewise-defined function; so let’s integrate \( dx \, dy \).]

The two curves intersect when \( x^2 = 6 - x \), which is to say \( x^2 + x - 6 = 0 \), i.e., \( (x - 2)(x + 3) = 0 \), so at \( x = 2 \) and \( x = -3 \). A sketch of the region, however, shows that the region in question is in the first quadrant, and hence the point of intersection has \( x \)-coordinate \( x = 2 \). However, we actually need the \( y \)-coordinate, which is \( y = 2^2 = 6 - 2 = 4 \).

Since the curve \( y = x^2 \), which along the edge of this region is \( x = \sqrt{y} \), is the left boundary of the region, and \( y = 6 - x \), which is \( x = 6 - y \), is the right boundary, the integral is

\[
\int_0^4 \int_0^{6-y} x^2 + y \, dy \, dx = \int_0^4 \frac{x^3}{3} + xy \bigg|_{y=x^2}^{y=6-y} \, dx = \int_0^4 \frac{1}{3} \left( (6-y)^3 - y^{3/2} \right) + (6-y)y - y^{3/2} \, dy
\]

\[
= \int_0^4 \frac{1}{3} (6-y)^3 - \frac{4}{3} y^{3/2} + 6y - y^2 \, dy = \frac{1}{12} (6-y)^4 - \frac{8}{15} y^{5/2} + 3y^2 - \frac{1}{3} y^{3/2} \bigg|_0^1
\]

\[
= -6 \cdot \frac{12}{15} - 8 \cdot \frac{32}{15} + 48 \cdot \frac{64}{3} + 6^4 \cdot \frac{12}{12} = -68 - \frac{256}{15} + 48 + 3 \cdot 6^2 = -\frac{596}{15} + \frac{156}{15} = \frac{1744}{15}.
\]

7. Let \( D \) be the region in the plane that lies inside the circle \( x^2 + y^2 = 6y \) but outside the circle \( x^2 + y^2 = 9 \). Compute the area of \( D \).

**Answer.** The first circle is \( r^2 = 6 \sin \theta \), i.e., \( r = 6 \sin \theta \); and the second is \( r = 3 \). They intersect when \( 3 = 6 \sin \theta \), i.e., \( \sin \theta = 1/2 \), so \( \theta = \pi/6 \) and \( \theta = 5\pi/6 \). [And the region is definitely in the sector between these two angles, since the first circle is completely above the \( x \)-axis.] So the area is

\[
\int_{0}^{5\pi/6} \int_{\pi/6}^{\sin \theta} r \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{2} r^2 \bigg|_{r=3}^{r=\sin \theta} \, d\theta = \int_{\pi/6}^{5\pi/6} 18 \sin^2 \theta - \frac{9}{2} \, d\theta
\]

\[
= \int_{\pi/6}^{5\pi/6} 9 \sin 2\theta \, d\theta = \frac{9}{2} \int_{\pi/6}^{\pi/2} \sin 2\theta \bigg|_{\pi/6}^{\pi/2} = \left( \frac{15\pi}{4} + \frac{9\sqrt{3}}{4} \right) - \left( \frac{3\pi}{4} - \frac{9\sqrt{3}}{4} \right) = 3\pi + \frac{9\sqrt{3}}{2}.
\]

8. Compute \( \int_0^1 \int_{x^{1/3}} \cos(\pi y^4/2) \, dy \, dx \).

**Answer.** The first integration looks hopeless, so we note that the region of integration is bounded on the left by \( x = 0 \), above by \( y = 1 \), and on the right (and below) by \( y = x^{1/3} \), which is equivalently \( x = y^3 \). The lowermost point of the region is the origin. Thus, the integral is

\[
\int_0^1 \int_0^{y^3} \cos(\pi y^4/2) \, dx \, dy = \int_0^1 x \cos(\pi y^4/2) \bigg|_{x=0}^{x=y^3} \, dy = \int_0^1 y^3 \cos(\pi y^4/2) \, dy
\]

\[
[u = \pi y^4/2, du = 2\pi y^3 \, dy] = \frac{1}{2\pi} \int_0^{\pi/2} \cos u \, du = \frac{1}{2\pi} \sin u \bigg|_0^{\pi/2} = \frac{1}{2\pi}(1 - 0) = \frac{1}{2\pi}.
\]

9. Find the volume of solid that lies between the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - 3x^2 - 3y^2 \).
10. Find the volume of the solid bounded by the surface \( z = e^x \) and the planes \( x = y, x = 1, y = 0, \) and \( z = 0. \)

**Answer.** Squashing onto the \( xy \)-plane gives the triangle bounded by \( x = y, x = 1, \) and \( y = 0. \) So the volume is

\[
\int_0^1 \int_0^x e^y \, dy \, dx = \int_0^1 \left[ e^x \right]_0^1 = e - 1.
\]

11. Let \( E \) be the tetrahedron with vertices \((0, 0, 0), (1, 0, 0), (0, 2, 0), \) and \((0, 0, 4), \) and with constant density 1. Find the mass and center of mass of \( E. \)

**Answer.** The four faces of the tetrahedron are the coordinate planes and the plane through the last three points. Taking the vectors between pairs of these points, we see that \( \langle -1, 0, 4 \rangle \) and \( \langle 0, -2, 4 \rangle \) lie in this plane. Taking the cross product shows that \( \langle 8, 4, 2 \rangle, \) and hence \( \langle 4, 2, 1 \rangle, \) is normal to this plane. So the plane has equation \( 4x + 2y + (z - 4) = 0, \) i.e., \( z = 4 - 4x - 2y. \) Squashing the tetrahedron onto the \( xy \)-plane gives the triangle bounded by \( x = 0, y = 0, \) and \( y = 2 - 2x. \) Thus, the mass is

\[
m = \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} dz \, dy \, dx = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y) dy \, dx = \int_0^1 (4 - 4x) y - y^2 \big|_{y=0}^{y=2-2x} \, dx
\]

Next, the \( yz \)-moment is

\[
\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} x \, dy \, dx = \int_0^1 \int_0^{2-2x} 4x - 4x^2 - 2xy \, dy \, dx
\]

The \( xz \)-moment is

\[
\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} y \, dy \, dx = \int_0^1 \int_0^{2-2x} 4y - 4xy - 2y^2 \, dx \, dy = \int_0^1 (2 - 2x)y^2 - \frac{2}{3} y^3 \big|_{y=0}^{y=2-2x} \, dx
\]

Last, the \( yx \)-moment is

\[
\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} z \, dz \, dy \, dx = \frac{1}{2} \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)^2 \, dy \, dx
\]

Therefore, the mass is \( 4/3, \) and the center of mass is \( \left\{ \frac{3}{4}, \frac{2}{3}, \frac{4}{3} \right\} = \left\{ \frac{1}{4}, \frac{1}{2}, 1 \right\}. \)

12. Let \( E \) be the solid region below the plane \( z = 10 - x, \) above the plane \( z = x + y, \) and between the surfaces \( x = y^2 \) and \( y = x - 2. \) Compute \( \iiint_E y \, dV. \)
Answer. The last two surfaces are vertical and give the region in the \(xy\)-plane with the (sideways) parabola \(x = y^2\) to the left, and the line \(x = y + 2\) to the right. (The intersection points are \((4, 2)\) and \((1, -1)\).) Meanwhile, the two original planes intersect above the line \(y = 10 - 2x\), which, as can be seen from a sketch, is well outside the region just described. (That is, the two planes don’t further cut down the region in the \(xy\)-plane.) So if we squash the solid to the \(xy\)-plane, we get the 2D region between \(x = y^2\) and \(x = y + 2\) that we just described. So the integral is

\[
\iiint_E y\,dV = \int_{-1}^{2} \int_{y^2}^{10-x} y\,dz\,dx\,dy = \int_{-1}^{2} \int_{y^2}^{10-x} (10 - 2x - y)\,y\,dx\,dy
\]

\[
= \int_{-1}^{2} (10x - x^2 - xy)\,dy = \int_{-1}^{2} (10y + 20 - y^2 - 4y - 4y^2 - 2y - 10y^2 + y^4 + y^3)\,dy
\]

\[
= \int_{-1}^{2} y^5 + y^4 - 12y^3 + 4y^2 + 16y\,dy = \frac{y^6}{6} + \frac{y^5}{5} - 3y^4 + \frac{4y^3}{3} + 8y^2\bigg|_{-1}^{1} = 32 + 32 - 48 + 32 - \frac{1}{6} + \frac{1}{5} + 3 + \frac{4}{3} - 8 = \frac{68}{3} + \frac{33}{5} - \frac{1}{6} - 21 = 81\quad[10]
\]

13. Let \(E\) be the solid region between the parabolic cylinder \(y = 2z^2\) and the plane \(y = -2z\), bounded in front by the plane \(x = 4\) and in back by the plane \(x + z = y\). Compute \(\iiint_E z^2\,dV\).

Answer. Given that the first two surfaces described are both cylinders running parallel to the \(x\)-axis, it makes sense to squash the solid onto the \(yz\)-plane. In the \(yz\)-plane, the curve \(y = 2z^2\) [which is a parabola opening along the positive \(y\)-axis] and \(y = -2z\) enclose a region in the third quadrant, extending diagonally from the origin down to the point \((y, z) = (2, -1)\) where the two curves intersect. In front of this region, the plane \(x = 4\) always lies in front of the plane \(x + z = y\) [because, in particular, \(x = y - z\) attains its maximum value on the region at the corner \((2, -1, 3) \leq 4\)] Therefore, the integral is

\[
\iiint_{-1}^{2} \int_{y-z}^{4} \int_{y^2}^{z} z^2\,dx\,dy\,dz = \int_{-1}^{2} \int_{y-z}^{4} \left[ xz^2\right]_{y-z}^{z} \,dy\,dz = \int_{-1}^{2} \int_{y-z}^{4} 4z^2 - yz^2 + z^3 \,dy\,dz
\]

\[
= \int_{-1}^{2} \left[ 4yz^2 - \frac{1}{2}y^2z^2 + yz^3\right]_{y-z}^{2z} \,dz = \int_{-1}^{2} -8z^3 - 8z^4 - 2z^5 - 2z^4 - 2z^5 \,dz
\]

\[
= \int_{-1}^{2} 12z^4 - 2z^5 - 2z^6 + z^7\,dz = -12 \cdot \frac{2}{5} + \frac{1}{3} + \frac{2}{7} = 2 - \frac{12}{5} + \frac{3}{2}
\]

As before, on an exam, leave a messy sum of fractions like that; but for the record, this one is 23/105.

14. Let \(E\) be the solid lying inside the sphere \(x^2 + y^2 + z^2 = 4\) and below the cone \(z = \sqrt{x^2 + y^2}\). Compute \(\iiint_E z^2\,dV\).

Answer. In spherical coordinates, the sphere is \(\rho = 2\), and the cone is \(\phi = \pi/4\). The solid \(E\) is the full ball with the portion above the cone removed. Thus,

\[
\iiint_E z^2\,dV = \int_{0}^{2\pi} \int_{\pi/4}^{\pi} \int_{0}^{2} (\rho \cos \phi)^2 \rho^2 \sin \phi \,d\rho \,d\phi \,d\theta = \left[ \int_{0}^{2\pi} d\theta \right] \left[ \int_{\pi/4}^{\pi} \cos^2 \phi \sin \phi \,d\phi \right] \left[ \int_{0}^{2} \rho^4 \,d\rho \right]
\]

and substituting \(u = \cos \phi\), \(du = -\sin \phi \,d\phi\) in the second integral, we get

\[
(2\pi - 0) \left( -\int_{\sqrt{2}/2}^{1} u^2\,du \right) \left( \frac{1}{2} \right) = 2\pi \left( \frac{u^3}{3} \right) \bigg|_{-\sqrt{2}/2}^{1} = 2\pi \left( \frac{1}{3} \right) = 64\pi / 15 \left( \frac{2\sqrt{2}}{8} - (-1) \right) = 16\pi / 15 (\sqrt{2} + 4)
\]

15. Let \(E\) be the portion of the first octant bounded by the \(xz\)-plane, the \(yz\)-plane, the plane \(z = 4\), and the paraboloid \(z = x^2 + y^2\). Compute \(\iiint_E x\,dV\).

Answer. \(z = x^2 + y^2\) is a circular paraboloid (i.e., bowl) opening upward from the origin. So with the plane \(z = 4\), we get a filled-in bowl of height 4. (The paraboloid is the bottom, and the plane is
the top.) Since \( E \) is in the first octant, \( E \) is just the front right quarter of this solid bowl, with the back and left faces being the \( yz \)- and \( xz \)-planes.

Let’s use cylindrical coordinates, so that the paraboloid becomes \( z = r^2 \), and the plane remains \( z = 4 \). The two surfaces intersect when \( r^2 = 4 \), i.e., \( r = 2 \). So the projection of \( E \) onto the \( xy \)-plane is the quarter disk (in the first quadrant) of radius 2 centered at the origin. The integral is therefore

\[
\begin{align*}
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (r \cos \theta) r \, dz \, dr \, d\theta & = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 r^2 \cos \theta \, dz \, dr \, d\theta \\
& = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (4r^2 - r^4) \cos \theta \, dr \, d\theta \\
& = \int_0^{\pi/2} \int_0^{\pi/2} \left( \frac{4}{3} r^3 - \frac{1}{5} r^5 \right) \cos \theta \, d \theta \\
& = \int_0^{\pi/2} \left( \frac{32}{3} - \frac{32}{5} \right) \cos \theta \, d \theta = 32 \left( \frac{1}{3} - \frac{1}{5} \right) (1 - 0) = 32 \left( \frac{1}{3} - \frac{1}{5} \right) = \left( \frac{64}{15} \right).
\end{align*}
\]

16. Let \( E \) be the solid region between the spheres \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 2 \), and inside the (upper half of the double) cone \( z = \sqrt{x^2 + y^2} \). The density of \( E \) at \( (x, y, z) \) is \( z^2 \). Compute the mass of \( E \).

**Answer.** In spherical coordinates, the spheres cut out \( 1 \leq \rho \leq \sqrt{2} \), and the cone forces \( 0 \leq \phi \leq \pi/4 \).

Recalling \( z = \rho \cos \phi \), the mass is

\[
\begin{align*}
\int_0^{\pi/2} \int_0^{\sqrt{2}} \int_0^{\rho = 1} \rho^2 \rho \sin \phi \, d\phi \, d\rho \, d\theta & = 2\pi \left( \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \right) \left( \int_0^{\sqrt{2}} \rho^4 \, d\rho \right) \\
& = -2\pi \left( \frac{\sqrt{2}}{1} \right) \left( \int_1^2 \frac{u^3}{u} \, du \right) \left( \int_0^{\pi/2} \cos \phi \, d\phi \right) = -2\pi \left( \frac{\sqrt{2}}{1} \right) \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\pi}{2} \right) = \frac{\pi}{30} (4\sqrt{2} - 1) (4 - \sqrt{2}).
\end{align*}
\]

17. Let \( a > 0 \). Find the volume of the solid inside both the sphere \( x^2 + y^2 + z^2 = a^2 \) and the cylinder \( x^2 + y^2 = ax \), and in the first octant.

**Answer.** [Note: it’s perhaps tempting to use spherical coordinates here, but the cylinder will make that a mess, so I’ll use cylindrical.]

Converting to cylindrical coordinates, the equation for the cylinder is \( r^2 = ar \cos \theta \), i.e. \( r = a \cos \theta \); note that is swept out for only \( 0 \leq \theta \leq \pi/2 \). [It’s not 0 to \( \pi/2 \) or 0 to 2\( \pi \).] Meanwhile, the sphere is \( r^2 + z^2 = a^2 \), i.e., \( z = \pm \sqrt{a^2 - r^2} \); but because we’re in the first octant, actually \( z \) goes from 0 to \( +\sqrt{a^2 - r^2} \). So the volume is

\[
\begin{align*}
\int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2 - r^2}} r \, dz \, dr \, d\theta & = \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} \, dr \, d\theta \quad [u = a^2 - r^2, du = -2r \, dr] \\
& = -\frac{1}{2} \int_0^{\pi/2} \int_0^{a \cos \theta} u^{3/2} \, du \, d\theta = -\frac{1}{2} \int_0^{\pi/2} u^{3/2} \, du \bigg|_{0}^{a \cos \theta} = \frac{a^3}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right),
\end{align*}
\]

18. Let \( E \) be the solid bounded by the surfaces \( y = 1 - x^2 \), \( y = 0 \), \( z = 1 \). Suppose that the density of the solid is \( \rho(x, y, z) = 2(1 + x^4) \). Compute the mass of \( E \).

**Answer.** The projection onto the \( xy \)-plane is the region \( D \) lying between the parabola \( y = 1 - x^2 \) and the line \( y = 0 \); so that’s a lump resting on the \( x \)-axis, from \(-1 \) to \( 1 \), with its peak at \((0,1)\). Note that \( y \leq 1 \) at every point of \( D \), so that the surface \( z = y \) lies below \( z = 1 \) over each point \((x, y)\) of \( D \).

Thus, the mass is

\[
\begin{align*}
\int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2 - r^2}} \rho \, dz \, dr \, d\theta & = \int_{-1}^{1} \int_0^{1-x^2} 2(1 + x^4) \, dy \, dx = \int_{-1}^{1} \int_0^{1-x^2} 2(1 + x^4) \bigg|_{z=y} \, dy \, dx \\
& = \int_{-1}^{1} \int_0^{1-x^2} 2(1 + x^4) \, dy \, dx \\
& = \int_{-1}^{1} 2(1 + x^4) \, dx \\
& = \left[ x + \frac{2}{5} x^5 \right]_{-1}^{1} = 2 \left( 1 + \frac{2}{5} \right) = \frac{14}{5}.
\end{align*}
\]
\[\int_{-1}^{1} \int_{0}^{1-x^2} 2(1+x^4)(1-y) \, dy \, dx = \int_{-1}^{1} (1+x^4)(2y-y^2) \bigg|_{y=0}^{1-x^2} \, dx = \int_{-1}^{1} (1+x^4)(2-2x^2-(1-x^2)^2) \, dx \]

\[\int_{-1}^{1} (1+x^4)(1-2x^2+2x^2-x^4) \, dx = \int_{-1}^{1} (1+x^4)(1-x^4) \, dx = \int_{-1}^{1} 1-x^8 \, dx \]

\[= x - \frac{x^9}{9} \bigg|_{1}^{1} = \left(1 - \frac{1}{9}\right) - \left(-1 + \frac{1}{9}\right) = \frac{8}{9} + \frac{8}{9} = \frac{16}{9}.\]

19. Compute \(\int_{-3}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{0}^{-\sqrt{9-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} \, dz \, dy \, dx.\)

**Answer.** The region of integration is the front half of the sphere of radius 3 centered at the origin; that is, the hemisphere with \(x \geq 0.\) Converting to spherical coordinates, recalling that \(y = \rho \sin \phi \sin \theta\) and \(\rho = \sqrt{x^2+y^2+z^2},\) the integral is

\[\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{0}^{3} \rho^5 \sin^3 \phi \sin^2 \theta \, d\rho \, d\phi \, d\theta = \left(\frac{1}{2} \int_{-\pi/2}^{\pi/2} \int_{0}^{\pi/2} 1 - \cos 2\theta \, d\theta\right) \left(\int_{0}^{\pi} \sin \phi (1 - \cos^2 \phi) \, d\phi\right) \left(\frac{1}{6} \rho^6 \bigg|_{0}^{3}\right)\]

\[= \frac{\theta}{2} - \frac{1}{4} \sin 2\theta \bigg|_{-\pi/2}^{\pi/2} \left(-\cos \phi + \frac{1}{3} \cos^3 \phi \bigg|_{0}^{\pi}\right) \left(\frac{243}{2}\right) = \left(\frac{\pi}{4} + \frac{\pi}{4} \left(\frac{2}{3} + \frac{2}{3}\right) \left(\frac{243}{2}\right)\right) = 81\pi.\]

20. Let \(E\) be the solid between the spheres \(x^2+y^2+z^2=1\) and \(x^2+y^2+z^2=9,\) below the cone \(z = \sqrt{x^2+y^2},\) and above the cone \(z = -\sqrt{3x^2+3y^2}.\) Compute \(\iiint_{E} x^2 \, dV.\)

**Answer.** We use spherical coordinates. The first cone is \(z = r,\) which means \(\phi = \pi/4;\) the second is \(z = -\sqrt{3r},\) which means \(\phi = 5\pi/6,\) since that is the unique angle between 0 and \(\pi\) for which \(\tan(\phi) = -1/\sqrt{3}.\) [Note that tan \(\phi = r/z.\)] Meanwhile, \(\theta\) goes all the way around, from 0 to \(2\pi,\) and \(\rho\) goes from the inner radius of 1 to the outer radius of 3. Thus, the integral is

\[\int_{0}^{2\pi} \int_{0}^{\pi/6} \int_{1}^{3} (\rho \sin \phi \cos \theta)^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_{0}^{2\pi} \cos^2 \theta \, d\theta \int_{0}^{\pi/4} \sin^3 \phi \, d\phi \int_{1}^{3} \rho^4 \, d\rho \]

\[= \frac{1}{2} \int_{0}^{2\pi} (1 + \cos 2\theta) \, d\theta \int_{0}^{\pi/6} \left(1 - \cos^2 \phi\right) \sin \phi \, d\phi \left[\phi^2 \bigg|_{0}^{\pi/6}\right] \]

\[= \frac{1}{2} \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta\right]^{2\pi}_{0} \cdot \int_{0}^{\pi/6} \left(1 + \sqrt{2}\right) \, du = \frac{242}{10} \cdot [2\pi - 0 + 0 - 0] \cdot \left[u^3 - u\right]^{\sqrt{3}/2}_{0}\]

\[= 242\pi \left[-\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 242\pi \left(\frac{120(10\sqrt{2} + 9\sqrt{3})}{60}\right) = \frac{121\pi}{60} (10\sqrt{2} + 9\sqrt{3}).\]

21. Let \(E\) be the solid between the paraboloid \(z = x^2 + y^2\) and the cone \(z = 2\sqrt{x^2+y^2}\) in the first octant. Compute \(\iiint_{E} y^2 \, dz.\)

**Answer.** Converting to cylindrical coordinates, the two surfaces are \(z = 2r\) and \(z = r^2.\) The paraboloid lies under the cone, until they intersect along the circle \(r = 2, z = 4.\) [Well, really for \(r^2 = 2r,\) so at both \(r = 0,\) which is at the origin, and at \(r = 2,\) which gives \(z = 4.\)] Meanwhile, being in the first octant, \(\theta\) runs from 0 to \(\pi/2.\) So in cylindrical coordinates, the integral is

\[\int_{0}^{\pi/2} \int_{0}^{2\pi} \int_{0}^{2r} (r \sin \theta)^2 rz \, dr \, d\theta = \int_{0}^{\pi/2} \int_{0}^{2r} r^3 \sin^2 \theta \, dz \, dr \, d\theta \]

\[= \int_{0}^{\pi/2} \int_{0}^{2r} \frac{1}{2} r^3 z^2 \sin^2 \theta \, dz \, dr \, d\theta = \int_{0}^{\pi/2} \int_{0}^{2r} (2r^5 - \frac{1}{2} r^7) \sin^2 \theta \, dr \, d\theta \]

\[= \int_{0}^{\pi/2} \frac{1}{2} (1 - \cos 2\theta) \, d\theta \left[\frac{1}{3} r^6 - \frac{1}{16} r^8\right]_{0}^{2} = \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta\right]_{0}^{\pi/2} \cdot \left(\frac{64}{3} - 16\right) \]

\[= \left(\frac{\pi}{4} - 0 - 0 + 0\right) \cdot \frac{16}{3} = \frac{4}{3}.\]