Math 211, Section 01, Fall 2019

Solutions to the Final Exam

1. (12 points) Let \( f(x, y) = \ln(2x + y) \).
   (a) Write an equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((-1, 3, 0)\).
   (b) Use your tangent plane equation from part (a) to estimate \( f(-1.1, 2.9) \).

   **Solution.** (a): We have \( f(-1, 3) = \ln(-2 + 3) = \ln(1) = 0 \). [Note: that’s implicit in the problem’s mention of the point \((-1, 3, 0)\).] We also have \( f_x(x, y) = 2(2x + y)^{-1} \) and \( f_y(x, y) = (2x + y)^{-1} \), so \( f_x(-1, 3) = 2(1)^{-1} = 2 \) and \( f_y(-1, 3) = (1)^{-1} = 1 \).

   Thus, the tangent plane is given by \( z = f(-1, 3) + f_x(-1, 3)(x + 1) + f_y(-1, 3)(y - 3) \), i.e., \( z = 2(x + 1) + (y - 3) \).

   (b): Using the tangent plane as an approximation, we have \( f(-1.1, 2.9) \approx 2(-1.1 + 1) + (2.9 - 3) = 2(-0.1) + (-0.1) = -0.3 \)

2. (18 points) Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function \( f(x, y) = x^2y - 2y^2 - 4y \).

   **Solution.** Compute \( f_x = 2xy \) and \( f_y = x^2 - 4y - 4 \). Setting \( f_x = 0 \) gives \( x = 0 \) or \( y = 0 \). If \( x = 0 \), then \( f_y = 0 \) gives \(-4y - 4 = 0 \), so \( y = -1 \), yielding the point \((0, -1)\). If \( y = 0 \), then \( f_y = 0 \) gives \( x^2 - 4 = 0 \), so \( x = \pm 2 \), yielding the two points \((\pm 2, 0)\).

   Next we compute \( f_{xx} = 2y \), \( f_{xy} = 2x \), and \( f_{yy} = -4 \). At the point \((0, -1)\), then,
   \[
   D = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = 8 - 0 > 0, \text{ with } f_{xx} = -2 < 0, \text{ i.e., local max.}
   \]

   At the points \((\pm 2, 0)\),
   \[
   D = \begin{vmatrix} 0 & \pm 4 \\ \pm 4 & -4 \end{vmatrix} = 0 - 16 < 0, \text{ i.e., saddles.}
   \]

   Thus, \( f \) has a local max at \((0, -1)\), and saddle points at \((-2, 0) \) and \((2, 0)\).

3. (12 points) Find an equation for the plane that contains the point \((0, 3, 0)\) and the line \( \vec{r}(t) = \langle 4 - t, 1 + 2t, 3t \rangle \).

   **Solution.** Let \( \vec{v} = \langle -1, 2, 3 \rangle \), which points parallel to the line. Let \( P = \vec{r}(0) = (4, 1, 0) \), which is on the line, and let \( Q = (0, 3, 0) \). Define \( \vec{w} = \overrightarrow{PQ} = \langle 4, -2, 0 \rangle \), and define
   \[
   \vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ 4 & -2 & 0 \end{vmatrix} = \langle 0 + 6, 12 - 0, 2 - 8 \rangle = (6, 12, -6).
   \]

   Since \( \vec{v} \) and \( \vec{w} \) both lie in the plane, then \( \vec{n} \) must be normal to the plane. Using \( \vec{n} \) and the point \( Q \), the plane is given by
   \[
   6(x - 0) + 12(y - 3) - 6(z - 0) = 0, \text{ i.e., } x + 2(y - 3) - z = 0, \text{ i.e., } x + 2y - z = 6
   \]

4. (18 points) Let \( f(x, y) = \begin{cases} 2x^3 + xy - y^2 & \text{if } (x, y) \neq (0, 0), \\ x^2 + y^2 & \text{if } (x, y) = (0, 0). \end{cases} \)

   (a) Prove that \( f \) is not continuous at \((0, 0)\).

   (b) Compute the directional derivative \( D_{\vec{u}}f(0, 0) \), where \( \vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \).

   **Solution.** (a): Approaching \((0, 0)\) along \( x = 0 \), we have...
\[ \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{0 + 0 - y^2}{0 + y^2} = \lim_{y \to 0} -1 = -1 \neq f(0, 0), \]
so \( f \) is not continuous at \((0, 0)\).

(b): Since \((0, 0) + h \mathbf{v} = \left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right)\), we have \( D_{x}f(0, 0) = \lim_{h \to 0} \frac{f(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}) - f(0, 0)}{h} \)
\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{h^3}{2\sqrt{2}} + \frac{2h^2}{2\sqrt{2}} - \frac{2h^2}{2} - \frac{2h^2}{2}\right) = \lim_{h \to 0} \frac{h^3/(2\sqrt{2})}{h^3} = \lim_{h \to 0} \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \]

5. (20 points) Find the absolute maximum and absolute minimum values of the function
\[ f(x, y) = xy^2 - 2x^3 \]
on the circle \( x^2 + y^2 = 9 \).

Solution. Let \( g(x, y) = x^2 + y^2 \), and apply Lagrange Multipliers. We compute
\[ f_x = y^2 - 6x^2, \quad f_y = 2xy, \quad g_x = 2x, \quad g_y = 2y, \]
so we must solve
\[ y^2 - 6x^2 = 2\lambda x, \quad 2xy = 2\lambda y, \quad x^2 + y^2 = 9. \]
The second equation gives \( 2y(x - \lambda) = 0 \), so either \( y = 0 \) or \( \lambda = x \). If \( y = 0 \), then the third equation gives \( x^2 = 9 \), i.e., \( x = \pm 3 \), so we have the two points \((\pm 3, 0)\).
If \( \lambda = x \), then the first equation becomes \( y^2 = 8x^2 \), so the third equation becomes \( 9x^2 = 9 \), and hence \( x = \pm 1 \). Then \( y^2 = 8 \), so \( y = \pm 2\sqrt{2} \); so we have the four points \((\pm 1, \pm 2\sqrt{2})\).
Testing these six points:
\[ f(3, 0) = -2(3)^3 = -54, \quad f(-3, 0) = -2(-3)^3 = 54, \]
\[ f(1, \pm 2\sqrt{2}) = (1)(8) - 2(1)^3 = 8 - 2 = 6, \]
\[ f(-1, \pm 2\sqrt{2}) = -1(8) - 2(-1)^3 = -8 + 2 = -6 \]
Thus, the maximum is 54, and the minimum is -54 (attained at \((-3, 0)\) and \((3, 0)\), respectively).

6. (20 points) Let \( E \) be the solid lying inside the sphere \( x^2 + y^2 + z^2 = 4 \) and above the cone \( z = \sqrt{x^2 + y^2} \) in the first octant. Compute \( \iiint_E x \, dV \).

Solution. Using spherical coordinates, \( \iiint_E x \, dV = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^2 \rho^2 \sin \phi \cos \theta \sin \phi \, d\rho \, d\phi \, d\theta \)
\[ = \int_0^{\pi/2} \cos \theta \, d\theta \int_0^{\pi/4} \frac{1}{2} (1 - \cos 2\phi) \, d\phi \int_0^2 \rho^3 \, d\rho = \left( \sin \theta \right)^{\pi/2} \left( \frac{\phi}{2} - \frac{1}{4} \sin 2\phi \right)^{\pi/4} \left( \frac{\rho^4}{4} \right)^{\pi/4} \]
\[ = (1 - 0) \left[ \left( \frac{\pi}{8} - \frac{1}{4} \right) - (0 - 0) \right] (4 - 0) = \frac{\pi}{2} - 1 \]

7. (15 points) Let \( C \) be the straight line segment from \((1, 0, 0)\) to \((0, -3, 2)\).

Compute \( \int_C \vec{F} \cdot d\vec{r} \), where \( \vec{F}(x, y, z) = (2y, -xz, z^2) \).

Solution. Parametrize \( C \) by \( \vec{r}(t) = (1 - t)(1, 0, 0) + t(0, -3, 2) = (1 - t, -3t, 2t) \), for \( 0 \leq t \leq 1 \). Then \( \vec{r}'(t) = (-1, -3, 2) \). So
\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^1 \left( 2(-3t), -2t(1-t), 4t^2 \right) \cdot (-1, -3, 2) \, dt \]
\[ = \int_0^1 6t + 6t(1-t) + 8t^2 \, dt = \int_0^1 12t + 2t^2 \, dt = 6t^2 + \frac{2}{3}t^3 \bigg|_0^1 = \left( 6 + \frac{2}{3} \right) - (0 - 0) = \frac{6 + \frac{2}{3}}{3} \]
8. (18 points) Let $C$ be the curve in the plane consisting of the straight line segment from $(0,0)$ to $(1,1)$, then around the arc of the circle $x^2 + y^2 = 2$ counterclockwise to the point $(-\sqrt{2},0)$, and finally straight along the $x$-axis back to $(0,0)$, as in this picture:

\[ \int_C (x^2 + y^2) \, dx + 5xy \, dy. \]

**Solution.** Let $D$ be the region enclosed by $C$; note that $C$ is already positively oriented with respect to $D$. With $P = x^2 + y^2$ and $Q = 5xy$, we compute \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 5y - 2y = 3y \). Thus, by Green’s Theorem, and doing the double integral via polar coordinates,

\[
\begin{align*}
\int_C (x^2 + y^2) \, dx + 5xy \, dy &= \iint_D 3y \, dA = \int_0^{\pi/2} \int_0^\sqrt{2} (3r \sin \theta) r \, dr \, d\theta = \int_0^{\pi/4} \sin \theta \, d\theta \int_0^\sqrt{2} 3r^2 \, dr \\
&= \left( -\cos \theta \Big|_{\pi/4}^\pi \right) \left( r^3 \Big|_0^{\sqrt{2}} \right) = \left( -(-1) + \frac{\sqrt{2}^3}{2} \right)(2\sqrt{2}) = \frac{2\sqrt{2}^3}{2} + 2
\end{align*}
\]

9. (12 points) Let $\vec{F}(x, y, z) = \langle 4xy + \cos x, \sin x - 2y^2, z \sin x \rangle$.

(a) Compute div $\vec{F}$ and curl $\vec{F}$.

(b) Decide whether $\vec{F}$ is the gradient of something, or the curl of something, or both, or neither. (You do **not** need to find the things it may be gradient or curl of, but you do **need** to (briefly) explain your reasoning.)

**Solution.** (a): $\text{div} \vec{F} = 4y - \sin x + (-4y) + (\sin x) = 0$ and

\[
\text{curl} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
4xy + \cos x & \sin x - 2y^2 & z \sin x
\end{vmatrix} = \langle 0, -z \cos x, \cos x - 4x \rangle
\]

(b): Since $\text{div} \vec{F} = 0$ and the domain is all of $\mathbb{R}^3$ [no “avocado holes”], then $\vec{F}$ is the curl of something Since curl $\vec{F} \neq 0$, then $\vec{F}$ is **not** the gradient of anything

10. (15 points) Let $\vec{F}(x, y) = \langle 2xy + 6x^2, x^2 - y^3 \rangle$.

(a) Show that $\vec{F}$ is conservative by finding a potential function $f(x, y)$ for $\vec{F}$.

(b) Let $C$ be the curve parametrized by $\vec{r}(t) = \langle t^2 - 2, 4t - 2t^2 \rangle$, for $1 \leq t \leq 2$.

Compute $\int_C \vec{F} \cdot d\vec{r}$.

**Solution.** (a): We need $f(x, y)$ with $f_x = 2xy + 6x^2$ and $f_y = x^2 - y^3$. From $f_x$, we get $f = x^2y + 2x^3 + g(y)$ for some function $g(y)$, and hence $f_y = x^2 + g'(y)$. From $f_y$, it follows that $g'(y) = -y^3$, so we may choose $g(y) = -\frac{y^4}{4}$.

That is, let $f(x, y) = x^2y + 2x^3 - \frac{y^4}{4}$ and we have $\nabla f = \vec{F}$.

(b): $C$ runs from $\vec{r}(1) = (-1, 2)$ to $\vec{r}(2) = (2, 0)$. Thus, by the Fundamental Theorem of Line Integrals,
\[ \int_C \vec{F} \cdot d\vec{r} = f(2, 0) - f(-1, 2) = (0 + 16 - 0) - (2 - 2 - 4) = 20 \]

11. (20 points) Let \( S \) be the surface of the tetrahedron with vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0),\) and \((0, 0, 2)\), oriented outward.

Let \( \vec{G}(x, y, z) = (2x^2 - yz, x^4, y^3 - 2xz) \). Compute the flux \( \iint_S \vec{G} \cdot d\vec{S} \) of \( \vec{G} \) through \( S \).

**Solution.** Since \( S \) is a closed surface, we may use the Divergence Theorem. Call the enclosed solid \( E \). We compute \( \text{div} \vec{G} = 4x + 0 - 2x = 2x \).

The top plane of \( E \) (passing through \((1, 0, 0), (0, 1, 0),\) and \((0, 0, 2)\)) is \( z = 2 - 2x - 2y \). [This can be computed either by finding a normal vector as the cross product of two of the edges of this triangle, e.g. \( \vec{n} = (1, -1, 0) \times (1, 0, -2) \), or by writing the unknown plane as \( ax + by + cz = d \), plugging in each of the three points to get \( a = d, b = d, \) and \( 2c = d \), before setting \( d = 2 \) and solving for \( a, b, c \).]

The projection of \( E \) onto the \( xy \)-plane is the triangle with vertices \((0, 0), (1, 0), \) and \((0, 1)\); the hypotenuse of this triangle runs along the line \( y = 1 - x \). Then

\[
\iint_E \vec{G} \cdot d\vec{V} = \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} 2x \, dy \, dz \, dx = \int_0^1 \int_0^{1-x} 2x \, dz \, dx = \int_0^1 2x \, dx \, \left[ z \right]_0^{2-2x-2y} = \int_0^1 2x \left[ 2(1-x)^2 - (1-x)^2 \right] \, dx = \int_0^1 2x(1-x)^2 \, dx = \int_0^1 2x - 4x^2 + 2x^3 \, dx = x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 \bigg|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{5}{6}.
\]

12. (20 points) Let \( S \) be the portion of the elliptic paraboloid \( y = x^2 + z^2 \) to the left of the plane \( y = 1 \), oriented with normal vectors pointing to the left (i.e., towards the negative-\( y \) direction).

Let \( \vec{F}(x, y, z) = \langle x, 0, x \rangle \). Compute the flux \( \iint_S \vec{F} \cdot d\vec{S} \) of \( \vec{F} \) through \( S \).

**Solution.** [Note: We can’t use the Divergence Theorem because the surface isn’t closed.] Observe that projecting \( S \) onto the \( xz \)-plane gives the closed disk of radius 1, inspiring us to try \( x = r \cos \theta \) and \( z = r \sin \theta \). Then \( y = x^2 + z^2 \) becomes \( y = r^2 \). That is, inspired by (sideways) cylindrical coordinates, we parametrize \( S \) by

\[
\vec{r}(r, \theta) = \langle r \cos \theta, r^2, r \sin \theta \rangle \quad \text{for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq r \leq 1.
\]

Then \( \vec{r}_r = \langle \cos \theta, 2r, \sin \theta \rangle \), and \( \vec{r}_\theta = \langle -r \sin \theta, 0, r \cos \theta \rangle \). So

\[
\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta & 2r & \sin \theta \\ -r \sin \theta & 0 & r \cos \theta \end{vmatrix} = \langle 2r^2 \cos \theta, -r, 2r^2 \sin \theta \rangle,
\]

which points to the left, since its \( y \)-component is \(-r < 0\). Thus, our orientation is already correct, and hence

\[
\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 \vec{F}(\vec{r}(r, \theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) \, dr \, d\theta.
\]

Let \( \theta = \theta_r \), \( \cos \theta = \cos \theta_r \), \( \sin \theta = \sin \theta_r \). Then

\[
\int_0^1 \int_0^{2\pi} \langle r \cos \theta_r, 0, r \sin \theta_r \rangle \cdot \langle 2r^2 \cos \theta_r, -r, 2r^2 \sin \theta_r \rangle \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^3 \cos^2 \theta_r + r^3 \sin \theta_r \cos \theta_r \, dr \, d\theta = \left( \int_0^{2\pi} (1 + \cos 2\theta_r) + 2 \sin \theta_r \cos \theta_r \, d\theta_r \right) \left( \int_0^1 r^3 \, dr \right) = \left( \theta + \frac{1}{2} \sin 2\theta_r + \sin^2 \theta_r \right) \left( \frac{r^4}{4} \right) \bigg|_0^1 = \frac{\pi}{2}.
\]

**OPTIONAL BONUS A.** (2 points) Consider the circle \( C \) in the \( yz \)-plane given by the equation \((y - 3)^2 + z^2 = 1\). Let \( S \) be the surface formed by revolving \( C \) around the \( z \)-axis. (Note: \( S \) is called a torus.) Find a parametrization of the surface \( S \).
Solution. Imagining the circle $C$ in the $(r,z)$-plane, centered at $(3,0)$, we can parametrize $C$ by $r(t) = 3 + \cos t$ and $z = \sin t$, for $0 \leq t \leq 2\pi$. Here, $r$ and $z$ are the usual $r$ and $z$ of cylindrical coordinates. Thus, revolving $C$ around the $z$-axis throws in the parameter $\theta$, with $x = r \cos \theta$ and $y = r \sin \theta$, for $0 \leq \theta \leq 2\pi$. Plugging in $r = r(t)$ from above, we get the following parametric surface:

$$\vec{r}(t, \theta) = ((3 + \cos t) \cos \theta, (3 + \cos t) \sin \theta, \sin t), \quad \text{for} \ 0 \leq t \leq 2\pi \text{ and } 0 \leq \theta \leq 2\pi$$

[There are other correct ways to do this, but this is probably the simplest.]

OPTIONAL BONUS B. (2 points) Let $\vec{F} = (y^2 - xz, x^3 - yz, xyz + z^2)$. Find a vector field $\vec{G}$ such that $\text{curl} \vec{G} = \vec{F}$.

Solution. We seek 3-variable functions $P$, $Q$, $R$ such that

$$R_y - Q_z = y^2 - xz, \quad P_z - R_x = x^3 - yz, \quad \text{and} \quad Q_x - P_y = xy + z^2.$$ 

 Arbitrarily, looking at the first equation, let’s try hoping we can have $R_y = y^2$ and $Q_z = xz$, so $R = \frac{y^3}{3} + f(x, z)$ and $Q = \frac{xz^2}{2} + g(x, y)$, for some 2-variable functions $f(x, z)$ and $g(x, y)$.

Then $R_x = f_x$, so the second equation gives $P_z = x^3 - yz + f_x(x, z)$, and hence $P = x^3 z - \frac{yz^2}{2} + h(x, z)$, where $h$ is some 2-variable functions such that $h_z(x, z) = f_x(x, z)$.

Therefore, $Q_x = \frac{z^2}{2} + g_x$ and $P_y = -\frac{z^2}{2}$, so $Q_x - P_y = z^2 + g_x$. Thus, the third equation gives $g_x = xy$.

To sum up, our only restrictions on the three unknown two-variable functions $f(x, z)$, $g(x, y)$, and $h(x, z)$ are that $g_x(x, y) = xy$ and $f_x(x, z) = h_z(x, z)$. Let’s choose $f = h = 0$ and $g(x, y) = \frac{x^2 y}{2}$, which have these properties. That is, we are choosing $\vec{G} = (P, Q, R)$, where

$$P(x, y, z) = x^3 z - \frac{yz^2}{2}, \quad Q(x, y, z) = \frac{xz^2}{2} + \frac{x^2 y}{2}, \quad R(x, y, z) = \frac{y^3}{3}$$

Sure enough, these choices give $R_y - Q_z = y^2 - xz$, $P_z - R_x = x^3 - yz - 0$, and $Q_x - P_y = \frac{z^2}{2} + xy + \frac{z^2}{2} = xy + z^2$, so that $\text{curl} \vec{G} = \vec{F}$.

[Note: There are many, many correct ways to do this. In particular, for any $C^2$ function $f(x, y, z)$, we have $\text{curl}(\vec{G} + \nabla f) = \text{curl}(\vec{G}) + \nabla f = \vec{F} + \vec{0} = \vec{F}$. That is, adding any conservative vector field to our answer above will also give a correct answer!]

OPTIONAL BONUS C. (1 point) Earlier this week, a court in Pakistan convicted a former president of that nation of treason and sentenced him to death. Name this former president of Pakistan.