Math 211, Section 01 Fall 2019

**Solutions to the Fall 2017 Final Exam**

1. Find an equation for the plane that contains the line \( \vec{r}(t) = (2 + t, 3t, 1 - t) \) and the point \((2, 3, -1)\).
   **Solution.** The direction vector \( \vec{v} = (1, 3, -1) \) lies in the plane. Since the point \( \vec{r}(0) = (2, 0, 1) \) is on the plane, then the vector
   \[
   \vec{w} = (2, 3, -1) - (2, 0, 1) = (0, 3, -2)
   \]
   also lies in the plane. Taking the cross product, the plane has normal vector
   \[
   \vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix}
   \vec{i} & \vec{j} & \vec{k} \\
   1 & 3 & -1 \\
   0 & 3 & -2 
   \end{vmatrix} = (-6 + 3, 0 + 2, 3 - 0) = (-3, 2, 3)
   \]
   Using \( \vec{n} \) and the point \((2, 3, -1)\), then, the plane is given by
   \[
   -3(x - 2) + 2(y - 3) + 3(z + 1) = 0, \text{ i.e., } -3x + 2y + 3z = -3.
   \]

2. Let \( f(x, y) = 2 + \ln(3x + y^2) \). Write the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((-1, 2, 2)\), and then use it to estimate \( f(-1.2, 2.1) \).
   **Solution.** We compute \( f_x(x, y) = \frac{3}{3x + y^2} \) and \( f_y(x, y) = \frac{2y}{3x + y^2} \). So \( f_x(-1, 2) = 3 \) and \( f_y(-1, 2) = 2 \). Since we also have \( f(-1, 2) = 2 \), the tangent plane is given by \( z = 2 + 3(x + 1) + 4(y - 2) \).
   [If you wish, that’s \( z = 3x + 4y - 3 \); but the above form is best for what’s to come.] So we estimate \( f(-1.2, 2.1) \approx 2 + 3(-1.2 + 1) + 4(2.1 - 2) = 2 + 3(-0.2) + 4(0.1) = 2 - 0.6 + 0.4 = 1.8 \)

3. Let \( f(x, y) = \begin{cases} 2x^3 + 4xy - y^3 & \text{if } (x, y) \neq (0, 0), \\ 3x^2 + y^2 & \text{if } (x, y) = (0, 0). \end{cases} \)
   **Solution.** (a): Approaching along the line \( y = x \), we get
   \[
   \lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{2x^3 + 4x^2 - x^3}{4x^2} = \lim_{x \to 0} \frac{x + 4}{4} = 1,
   \]
   which does not equal \( f(0, 0) = 0 \), so \( f \) is indeed not continuous at \((0, 0)\). QED

   (b): \( f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{2h^3 + 0 - 0}{3h^2 + 0} - \lim_{h \to 0} \frac{0 + 0 - h^3}{h} = \lim_{h \to 0} \frac{2h^3}{3h^2} = \lim_{h \to 0} \frac{2}{3} = \frac{2}{3}. \)
   \[
   f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 + h^2 - 0}{0 + h^2} = \lim_{h \to 0} \frac{-h^3}{h^3} = \lim_{h \to 0} -1 = -1.
   \]

4. Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function \( f(x, y) = 6x^2 + 3y^2 - xy^2 + 9 \).
   **Solution.** We compute \( f_x = 12x - y^2 \) and \( f_y = 6y - 2xy \). Setting \( f_y = 0 \) gives \( 2y(3 - x) \), so either \( y = 0 \) or \( x = 3 \).
   If \( y = 0 \), then setting \( f_x = 0 \) yields \( x = 0 \), giving the point \((0, 0)\).
   If \( x = 3 \), then setting \( f_y = 0 \) yields \( y^2 = 36 \), giving the two points \((3, \pm 6)\).
   We compute \( f_{xx} = 12, f_{xy} = -2y, \) and \( f_{yy} = 6 - 2x \).
At \((0,0)\): \(D = 12(6) - 0^2 = 72 > 0\), and \(f_{xx} = 12 > 0\), so \(f\) has a local minimum at \((0,0)\).
At \((3,\pm 6)\): \(D = 12(0) - (-\pm 12)^2 = -144 < 0\), so \(f\) has saddle points at both \((3,6)\) and \((3,-6)\).

5. Find the maximum and minimum values of the function \(f(x, y) = x^2 + 8y\) on the ellipse \(x^2 + 2y^2 = 18\).

**Solution.** Let \(g(x, y) = x^2 + 2y^2\). Via Lagrange Multipliers, we solve \(\nabla f = \lambda \nabla g\) and \(g = 18\), i.e.,
\[
2x = 2x\lambda, \quad 8 = 4y\lambda, \quad x^2 + 2y^2 = 18.
\]
The first equation gives either \(x = 0\) or \(\lambda = 1\).
If \(x = 0\), then the third equation is \(2y^2 = 18\), so \(y^2 = 9\), i.e. \(y = \pm 3\). We get two points: \((0, \pm 3)\).
If \(\lambda = 1\), then the second equation is \(y = 2\), so the third equation is \(x^2 = 10\). We get two more points: \((\pm \sqrt{10}, 2)\).
Plugging these four points into \(f\) gives
\[
\begin{align*}
f(0, 3) &= 24, \quad f(0, -3) = -24, \quad f(\pm \sqrt{10}, 2) = 10 + 16 = 26.
\end{align*}
\]
Thus, the maximum value is 26, and the minimum is -24.

6. Let \(E\) be the solid bounded below by the surface \(z = y^2\), bounded above by the plane \(z = 1\), and bounded in the back and front by the planes \(y = x\) and \(x = 2\), respectively. Suppose that the density of \(E\) is given by \(\rho(x, y, z) = 10z\). Compute the mass of \(E\).

**Solution.** Projecting onto the \(yz\)-plane yields the region below the line \(z = 1\) and above the curve \(z = y^2\); so \(-1 \leq y \leq 1\), and \(y^2 \leq z \leq 1\).
The third variable \(x\) then runs from \(x = y\) (at the back) to \(x = 2\) (at the front). So the mass is
\[
\iiint_E \rho \, dV = \int_{-1}^1 \int_{y^2}^1 \int_{10z}^{10z(2-y)} 10z \, dx \, dz \, dy = \int_{-1}^1 \int_{y^2}^1 10z \, dx \, dz \, dy
= \int_{-1}^1 5z^2(2-y) \, dy = \int_{-1}^1 5(1-y^4)(2-y) \, dy = \int_{-1}^1 5y^5 - 5y - 10y^4 + 10y \, dy
= \frac{5y^6}{6} - \frac{5}{2}y^2 - 2y^5 + 10y\Big|_{-1}^1 = \left(\frac{5}{6} - \frac{5}{2} - 2 + 10\right) - \left(\frac{5}{6} - \frac{5}{2} + 2 - 10\right) = 8 + 8 = 16.
\]

7. Let \(E\) be the solid lying inside the sphere \(x^2 + y^2 + z^2 = 1\), above (i.e., inside) the cone \(z = \sqrt{x^2 + y^2}\), and inside the first octant. Compute \(\iiint_E 8z \, dV\).

**Solution.** In spherical coordinates, the integral becomes
\[
\iiint_E 8z \, dV = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\rho_1} 8(\cos\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\theta \int_0^{\rho_1} 8\rho^3 \, d\rho
\]
\[
[u = \sin\phi, du = \cos\phi \, d\phi] \quad \frac{\pi}{2} \left[ \int_0^{\sqrt{1/2}} u \, du \right] \left[ 2\rho^4 \right]_{\rho_0}^{\rho_1} = \frac{\pi}{4} \left[ \frac{u^2}{2} \right]_{0}^{\sqrt{1/2}} (2 - 0)
\]
\[
= \frac{\pi}{4} \left( \frac{1}{4} - 0 \right) (2) = \frac{\pi}{4}
\]

8. [NOTE: SKIP #8! NOT ON FALL 2018 EXAM!] Let \(C\) be the quarter-circle arc of radius 2 in the first quadrant of the \(xy\)-plane, as shown in the figure. Compute \(\int_C x^3 \, ds\).

**Solution.** [Included just for kicks.] Parametrize \(C\) by \(\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle\) for \(0 \leq t \leq \pi/2\), so \(\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle\), and hence \(||\vec{r}'(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2\).
Thus,
\[
\int_C x^3 \, ds = \int_0^{\pi/2} (2\cos t)^3(2) \, dt = 16 \int_0^{\pi/2} (1 - \sin^2 t) \cos t \, dt \quad [u = \sin t, du = \cos t \, dt]
\]
9. Let \( \vec{F}(x, y) = \langle 6x - 3x^2y, 6y^2 - x^3 \rangle \).

   9a. Show that \( \vec{F} \) is conservative by finding a potential function for \( \vec{F} \).

   9b. Let \( C \) be the quarter-circle path running from (2, 1) to (1, 2), counterclockwise along
   the arc of the circle \((x - 1)^2 + (y - 1)^2 = 1\), as shown below. Compute \( \int_C \vec{F} \cdot d\vec{r} \).

   ![Diagram of a quarter-circle path]

   Solution. (a): Seeking \( f(x, y) \) so that \( \nabla f = \vec{F} \), we solve \( f_x = 6x - 3x^2y \) to yield
   \( f = 3x^2 - x^3y + g(y) \).
   Taking \( \partial_y f_2 \), we get \( f_y = -3x^2 + g'(y) \). But since \( f_y = 6y^2 - x^3 \), we have \( g'(y) = 6y^2 \), so we may pick
   \( g(y) = 2y^3 \). That is, let \( f(x, y) = 3x^2 - x^3y + 2y^3 \). Sure enough, \( \nabla f = \vec{F} \).

   (b): By FTLI, \( \int_C \vec{F} \cdot d\vec{r} = \int_C (\nabla f) \cdot d\vec{r} = f(1, 2) - f(2, 1) = (3 - 2 + 16) - (12 - 8 + 2) = 17 - 6 = 11 \).

10. Let \( C \) be the boundary of the triangle with vertices (0, 0), (2, 1), and (0, 1), oriented counterclockwise, as shown in the figure. Let \( \vec{G}(x, y) = \langle xy^2, 3x^2y + \cos^8 y \rangle \).

   ![Diagram of a triangle]

   Compute \( \int_C \vec{G} \cdot d\vec{r} \).

   Solution. Let \( D \) be the region inside the triangle; note that \( C \) is positively oriented with respect to
   \( D \). Writing \( \vec{G} = \langle P, Q \rangle \), we have \( Q_x - P_y = 6xy - 2xy = 4xy \), and hence by Green’s Theorem,
   \[
   \int_C \vec{G} \cdot d\vec{r} = \iint_D 4xy \, dA = \int_0^1 \int_0^{2y} 4xy \, dx \, dy = \int_0^1 8y^3 \, dy = \frac{32}{4} = 2.
   \]

11. Let \( S \) be the closed surface consisting of the portion of the paraboloid \( z = 2 - 2x^2 - 2y^2 \) above the
   \( xy \)-plane, and the disk \( x^2 + y^2 \leq 1 \) in the \( xy \)-plane, oriented outward. Let \( \vec{G}(x, y, z) = \langle 2x^3, y^2z, -yz^2 \rangle \).
   Use the Divergence Theorem to compute the flux \( \iint_S \vec{G} \cdot d\vec{S} \) of \( \vec{G} \) through \( S \).

   Solution. We have \( \text{div} \vec{G} = 6x^2 + 2yz - 2yz = 6x^2 \). By the Divergence Theorem, we have
   \[
   \iint_S \vec{G} \cdot d\vec{S} = \iiint_E 6x^2 \, dV,
   \]
   where \( E \) is the solid enclosed by \( S \). In cylindrical coordinates, the bottom of \( E \) is
   \( z = 0 \), and the top is \( z = 2 - 2r^2 \). Moreover, the projection of \( E \) onto the \( xy \)-plane is the disk of
   radius 1, centered at the origin, since these two surfaces meet when \( 2 - 2r^2 = 0 \), i.e., \( r = 1 \). Thus,
   \[
   \iint_S \vec{G} \cdot d\vec{S} = \iiint_E 6x^2 \, dV = \int_0^{2\pi} \int_0^1 \int_{2r^2 - 12r^2}^{2r^2} 6r \cos^2 \theta \, dz \, d\theta \, dr = \frac{1}{2} \int_0^{2\pi} \cos^2 \theta \, d\theta \int_0^1 (1 + \cos 2\theta) \, d\theta \left( 3r^4 - 2r^6 \right)_{r=1}^{r=0} = \frac{1}{2} \left( (\pi - 0) - (0 - 0) \right) = \frac{\pi}{2}.
   \]

12. Let \( S \) be the portion of the surface \( z = x^2 + y^2 \) that lies in the first octant, and below the plane
   \( z = 4 \), oriented downward. Let \( \vec{F}(x, y, z) = \langle y, -x, xz \rangle \). Compute the flux \( \iint_S \vec{F} \cdot d\vec{S} \) of \( \vec{F} \) through \( S \).
Solution. The projection of $S$ onto the $xy$-plane is a quarter disk of radius 2, so I’ll use cylindrical coordinates as an inspiration. Parametrize $S$ by $\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle$, for $0 \leq \theta \leq \pi/2$, and $0 \leq r \leq 2$. We have $\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} 
\cos \theta & \sin \theta & 2r 
-r \sin \theta & r \cos \theta & 0
\end{vmatrix} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, r \rangle$,
which is oriented upward (because the third coordinate is $r > 0$, i.e., the wrong way. So instead we use the negative, $\vec{r}_\theta \times \vec{r}_r = \langle 2r^2 \cos \theta, 2r^2 \sin \theta, -r \rangle$. We have $\vec{F}(\vec{r}(r, \theta)) \cdot (\vec{r}_\theta \times \vec{r}_r) = \langle r \sin \theta, r \cos \theta, r^3 \cos \theta \rangle \cdot \langle 2r^2 \cos \theta, 2r^2 \sin \theta, -r \rangle = 2r^3 \sin \theta \cos \theta - 2r^3 \sin \theta \cos \theta - r^4 \cos \theta$. Thus, $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi/2} \int_0^2 -r^4 \cos \theta \, dr \, d\theta = \left[ -\sin \theta \right]_0^{\pi/2} \left[ \frac{r^5}{5} \right]_0^2 = (-1 - 0) \left( \frac{32}{5} - 0 \right) = -\frac{32}{5}.$