1. (15 points) Consider the integral \( \int_{-2}^{0} \int_{0}^{x^2} f(x, y) \, dy \, dx \)

(a). Sketch the region of integration.
(b). Rewrite the integral with the order of integration reversed.

**Answer.**
(a): We have \(-2 \leq x \leq 0\), and \(y\) running from the \(x\)-axis up to the curve \(y = x^2\), like so:

(b): The curve part of the region is \(x = -\sqrt{y}\), running from \((-2, 4)\) down to \((0, 0)\), so we get \(\int_{0}^{4} \int_{-\sqrt{y}}^{-2} f(x, y) \, dx \, dy\)

2. (20 points) Let \(D\) be the region in the plane inside the circle \(x^2 + y^2 = 2y\) (which is not centered at the origin), in the first quadrant, and above the line \(y = x\). Compute \(\iint_D 4xy \, dA\).

**Solution.** Here is the picture:

\[
\begin{align*}
\text{The region } D \text{ is the portion of the circle above the line in the first quadrant.}
\end{align*}
\]

In polar, the circle is \(r^2 = 2r \sin \theta\), so \(r = 2 \sin \theta\); and the line is \(\theta = \pi/4\). So
\[
\begin{align*}
\iint_D 4xy \, dA &= \int_{\pi/4}^{\pi/2} \int_{0}^{2 \sin \theta} 4(r \cos \theta)(r \sin \theta) r \, dr \, d\theta \\
&= \int_{\pi/4}^{\pi/2} \int_{0}^{2 \sin \theta} 4r^3 \cos \theta \sin \theta \, dr \, d\theta \\
&= \int_{\pi/4}^{\pi/2} \left. \frac{8}{3} u^6 \right|_{u=0}^{u=1} = \frac{8}{3} \left(1 - \frac{1}{8}\right) = \frac{8}{3} \left(\frac{7}{8}\right) = \frac{7}{3}
\end{align*}
\]

3. (20 points) Let \(E\) be the solid that is above (i.e., inside) the cone \(z = \sqrt{x^2 + y^2}\), in front of the \(yz\)-plane, and inside the sphere \(x^2 + y^2 + z^2 = 4\). Compute the mass of \(E\), if the density of \(E\) at the point \((x, y, z)\) is \(x\).

**Solution.** Let’s use spherical coordinates. We have \(-\pi/2 \leq \theta \leq \pi/2\), since we are in front of the \(yz\)-plane, Since the cone is \(\phi = \pi/4\), then being inside it means \(0 \leq \phi \leq \pi/4\). Thus, the mass is
\[
\begin{align*}
\iiint_E x \, dV &= \int_{\pi/4}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{0}^{4} (\rho \sin \phi \cos \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \left( \int_{0}^{\pi/2} \rho^3 \, d\rho \right) \left( \int_{0}^{\pi/2} \sin^2 \phi \, d\phi \right) \left( \int_{-\pi/2}^{\pi/2} \cos \theta \left(\frac{\rho^4}{4}\right) \right) \left( \int_{0}^{\pi/4} \frac{1}{2}(1 - \cos 2\phi) \, d\phi \right) \left( \sin \theta \right)_{-\pi/2}^{\pi/2}
\end{align*}
\]
4. (25 points) Let $E$ be the solid bounded by the surfaces $z = \sqrt{x}$, $z = x$, $y = 0$, and $y + 2z = 2$. Suppose the density of $E$ is given by $\rho(x, y, z) = x$. Take my word for it that the mass of $E$ is $1/20$. Compute the $z$-coordinate of the center of mass of $E$.

Solution. If we flatten $E$ onto the $xz$-plane, we get the region bounded by $z = \sqrt{x}$ and $z = x$,

which looks like this:

The two curves intersect at $(0, 0)$ and $(1, 1)$, so we have $0 \leq z \leq 1$ and $z^2 \leq x \leq z$. The last plane is $y = 2 - 2z$, which is always at least as large as $y = 0$; so $y = 0$ is the left side, and $y = 2 - 2z$ is the right. So the $xy$-moment is

\[
M_{xy} = \iiint_E z\rho(x, y, z)\,dV = \int_0^1 \int_0^z \int_0^{2z} xyz\,dy\,dz = \int_0^1 \int_0^z xz\,dy\,dz\]

\[
= \int_0^1 \int_0^z 2xz(1 - z)\,dy\,dz = \int_0^1 \int_0^z x^2z(1 - z)\,dz = \int_0^1 (z^2 - z^4)(z - z^2)\,dz
\]

\[
= \left. \frac{z^4}{4} - \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} \right|_0^1 = \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7}.
\]

Dividing by the mass $m = 1/20$, the $z$-coordinate of the center of mass is

\[
\bar{z} = 5 - 4 \cdot \frac{20}{6} + \frac{20}{7} = 1 - \frac{10}{3} + \frac{20}{7}.
\]

5. (20 points) Let $E$ be the solid in the first octant enclosed by the paraboloids $z = 2x^2 + 2y^2$ and $z = 3 - x^2 - y^2$. Compute $\iiint_E x^3\,dV$.

Solution. Let's use cylindrical coordinates. The first paraboloid is $z = 2r^2$, and the second is $z = 3 - r^2$. They intersect when $2r^2 = 3 - r^2$, so $3r^2 = 3$, so $r = 1$; and the second one is on top. We have $0 \leq \theta \leq \pi/2$ since $E$ is in the first octant. So the integral is

\[
\iiint_E x^3\,dV = \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{3 - r^2} (r\cos\theta)^3 r\,dz\,dr\,d\theta
\]

\[
= \int_0^{\pi/2} \int_0^1 (3r^4 - 3r^6)\cos^3\theta r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 zr^4\cos^3\theta \bigg|_{z=2r^2}^{z=3-r^2} dr\,d\theta
\]

\[
= \left( \frac{3}{5} - \frac{3}{7} \right) \int_0^{\pi/2} (1 - \sin^2\theta) \cos\theta d\theta
\]

\[
= \left( \frac{3}{5} - \frac{3}{7} \right) \int_0^1 (1 - u^2) du = \left( \frac{3}{5} - \frac{3}{7} \right) \left( u - \frac{u^3}{3} \right) \bigg|_0^1 = \left( \frac{3}{5} - \frac{3}{7} \right) \left( \frac{2}{3} - 0 \right) = \frac{2}{5} - \frac{2}{7}.
\]

OPTIONAL BONUS A. (2 points) Let $E$ be the solid in the first octant bounded by the coordinate planes and the two circular cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. Compute $\iiint_E y^2z^2\,dV$.

Solution. The first cylinder, when projected onto the $xy$-plane, becomes the vertical strip between $x = -1$ and $x = 1$; and the second becomes the horizontal strip between $y = -1$ and $y = 1$. So the
projection of the solid $E$ (being in the first octant) is the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The first cylinder $z = \sqrt{1 - x^2}$ is on top for $x \leq y$, and the second cylinder $z = \sqrt{1 - y^2}$ is on top for $y \leq x$. So to integrate over $E$, we must integrate separately over these two triangles in the $xy$-plane:

The lower triangle is $0 \leq x \leq 1$ with $0 \leq y \leq x$, with $0 \leq z \leq \sqrt{1 - x^2}$, and the upper triangle is $0 \leq y \leq 1$ with $0 \leq x \leq y$, with $0 \leq z \leq \sqrt{1 - y^2}$.

The integral on the first triangle is
\[
\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} y^2 z^2 \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{y^3}{3} \sqrt{1-x^2} \, dx = \int_0^1 \frac{x^3}{9} (1-x^2)^{3/2} \, dx
\]
\[
[u = 1 - x^2, \, du = -2x \, dx, \, x^2 = 1 - u] = \int_1^0 \frac{1}{18} (1-u)^{3/2} \, du = \frac{1}{18} \int_1^0 u^{3/2} - u^{3/2} \, du
\]
\[
= \frac{1}{18} \left( \frac{2}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right) \bigg|_1^0 = \frac{1}{18} \left( \frac{2}{7} \frac{2}{5} - \frac{2}{5} \frac{1}{7} \right) = \frac{1}{9} \left( \frac{1}{5} - \frac{1}{7} \right)
\]

The integral on the second triangle is
\[
\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} y^2 z^2 \, dz \, dx \, dy = \int_0^1 \int_0^y \frac{y^3}{3} \sqrt{1-y^2} \, dx \, dy
\]
\[
[u = 1 - y^2, \, du = -2y \, dy, \, y^2 = 1 - u] = \int_1^0 \frac{1}{6} (1-u)^{3/2} \, du = \frac{1}{6} \int_1^0 u^{3/2} - u^{3/2} \, du
\]
\[
= \frac{1}{6} \left( \frac{2}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right) \bigg|_1^0 = \frac{1}{6} \left( \frac{2}{7} \frac{2}{5} - \frac{2}{5} \frac{1}{7} \right) = \frac{1}{3} \left( \frac{1}{5} - \frac{1}{7} \right)
\]

Thus, the full integral is
\[
\int_0^1 \int_0^1 y^2 z^2 \, dV = \int_0^1 \int_0^x \frac{1}{9} \left( \frac{1}{5} - \frac{1}{7} \right) + \frac{1}{3} \left( \frac{1}{5} - \frac{1}{7} \right) = \frac{4}{9} \left( \frac{1}{5} - \frac{1}{7} \right) = \frac{8}{315}
\]

OPTIONAL BONUS B. (1 point) Name the capitals of two of the following three countries:
Ecuador, Ukraine, and Yemen.