1. (12 points) Show that  \( \lim_{(x,y) \to (0,0)} \frac{x^2y - 4xy}{x^2 + 5y^2} \) diverges.

**Solution.** Along the line \( y = 0 \) we have  \( \lim_{x \to 0} \frac{0 - 0}{x^2 + 0} = \lim_{x \to 0} 0 = 0 \), but along \( y = x \) we have  \( \lim_{x \to 0} \frac{x^3 - 4x^2}{x^2 + 5x^2} = \lim_{x \to 0} \frac{x - 4}{x + 5} = -\frac{2}{3} \neq 0 \), so the limit diverges.

2. (7 points) Let \( h(x, y) = x^2 - 3xe^y \). Compute the directional derivative \( D_{\vec{u}} h(1, 0) \), where \( \vec{u} \) is the unit vector in the direction of \( \vec{v} = \langle 4, -3 \rangle \).

**Solution.** Since \( ||\vec{v}|| = \sqrt{16 + 9} = 5 \), the unit vector is \( \vec{u} = \frac{1}{5} \langle 4, -3 \rangle \).

We compute \( \nabla h = \langle 2x - 3e^y, -3xe^y \rangle \). Since \( h \) is differentiable, we have  
\[
D_{\vec{u}} h(1, 0) = (\nabla h(1, 0)) \cdot \vec{u} = (2 - 3, -3(1)) \cdot \left( \frac{4}{5}, -\frac{3}{5} \right) = \frac{1}{5} \langle -1, -3 \rangle \cdot \langle 4, -3 \rangle = \frac{1}{5}(-4 + 9) = 1.
\]

3. (7 points) State the formal definition (i.e., the \( \varepsilon-\delta \) definition) of  \( \lim_{(x,y) \to (3,7)} 2x^2 + xy = -3 \).

**Answer.** For all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( (x, y) \) such that \( 0 < \sqrt{(x - 3)^2 + (y - 7)^2} < \delta \), we have \( |2x^2 + xy + 3| < \varepsilon \).

4. (20 points, 2 parts) Let \( f(x, y) = \begin{cases} 3y^3 - x^2y & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases} \)

(a). Compute \( f_x(0,0) \) and \( f_y(0,0) \).

(b). Compute \( D_{\vec{u}} f(0,0) \), where \( \vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle \).

**Solution.** (a): \( f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = \lim_{h \to 0} 0 = 0 \).

\[
f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{3h^3 - 0}{h} = \lim_{h \to 0} \frac{3h^3}{h} = \lim_{h \to 0} 3 = 3.
\]

(b): \( D_{\vec{u}} f(0,0) = \lim_{h \to 0} \frac{f(h/\sqrt{2}, -h/\sqrt{2}) - f(0,0)}{h/\sqrt{2}} = \lim_{h \to 0} \frac{-3h^3/(2\sqrt{2}) + h^3/(2\sqrt{2})}{h/\sqrt{2}} - 0 \)

\[
= \lim_{h \to 0} \frac{-2h^3/(2\sqrt{2})}{h/\sqrt{2}} = \lim_{h \to 0} \frac{-1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.
\]

5. (12 points) Let \( h(x, y) \) be a differentiable function such that 
\[
\nabla h(3, -1) = \langle -6, 5 \rangle, \quad \nabla h(-2, -3) = \langle 7, -2 \rangle, \quad \text{and} \quad \nabla h(1, -1) = \langle -2, 3 \rangle.
\]

Define \( f(s, t) = h(s + 5t^3, st) \). Compute \( f_s(3, -1) \).

**Solution.** Write \( z = h(x, y) = f(s, t) \), where \( x = s + 5t^3 \) and \( y = st \). Since all the functions involved are differentiable, we have 
\[
f_s = \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = h_x(x, y) \cdot (1) + h_y(x, y) \cdot (t).
\]

At \( (s, t) = (3, -1) \), we have \( x = 3 + 5(-1)^3 = -2 \) and \( y = (3)(-1) = -3 \). Thus, 
\[
f_s(3, -1) = h_x(-2, -3) \cdot 1 + h_y(-2, -3) \cdot (-1) = 7(1) + (-2)(-1) = 7 + 2 = 9.
\]
6. (20 points) Find the maximum and minimum values of the function \( f(x, y) = xy^2 \) on the ellipse \( 2x^2 + y^2 = 6 \).

**Solution.** Let \( g(x, y) = 2x^2 + y^2 \) and apply Lagrange multipliers. We have \( \nabla f = (y^2, 2xy) \) and \( \nabla g = (4x, 2y) \), so our equations are:
\[
\begin{align*}
2y^2 &= 4\lambda x, \\
2xy &= 2\lambda y, \\
2x^2 + y^2 &= 6.
\end{align*}
\]
The second equation gives \( 2y(x - \lambda) \), so either \( y = 0 \) or \( \lambda = x \).
If \( y = 0 \), then the third equation is \( x^2 = 3 \), so \( x = \pm \sqrt{3} \), giving the two points \((\pm \sqrt{3}, 0)\).
If \( \lambda = x \), the first equation is \( y^2 = 4x^2 \), so the third equation is \( 6x^2 = 6 \), so \( x = \pm 1 \). Thus, \( y^2 = 4 \), so \( y = \pm 2 \), with the signs independent. We get four more points: \((\pm 1, \pm 2)\).
We compute \( f(\pm \sqrt{3}, 0) = 0 \), \( f(1, \pm 2) = 4 \), \( f(-1, \pm 2) = -4 \), so the max is 4 and the min is -4.

7. (22 points) Find and classify (as local minimum, local maximum, or saddle point) all critical points of the function \( h(x, y) = 8x^3 - 6xy + y^3 + 2 \).

**Solution.** We compute \( \nabla h = (24x^2 - 6y, -6x + 3y^2) \). Solving \( \nabla h = 0 \), we have \( 24x^2 = 6y \) and \( 3y^2 = 6x \). The first equation gives \( y = 4x^2 \), and the second gives \( y^2 = 2x \). Plugging the first into the second yields \( 16x^4 = 2x \), so that \( 2x(8x^3 - 1) = 0 \). Thus, either \( x = 0 \) or \( x^3 = 1/8 \).
If \( x = 0 \), then the first equation gives \( y = 0 \). If \( x^3 = 8 \), then \( x = 1/2 \), and the first equation gives \( y = 1 \). Thus, we have two critical points: \((0, 0)\) and \((1/2, 1)\).
We have \( h_{xx} = 48x, h_{xy} = -6, h_{yy} = 6y \), so \( D = \begin{vmatrix} 48x & 6 \\ -6 & 6y \end{vmatrix} \).
At \((0, 0)\), we get \( D = \begin{vmatrix} 0 & -6 \\ -6 & 6 \end{vmatrix} = -36 < 0 \), so there is a saddle point at \((0, 0)\).
At \((1/2, 1)\), we get \( D = \begin{vmatrix} 24 & -6 \\ -6 & 6 \end{vmatrix} = 24(6) - 6(6) = 18(6) > 0 \), and \( h_{xx} = 24 > 0 \), so there is a local minimum at \((1/2, 1)\).

**OPTIONAL BONUS A. (2 points)** Prove that the function
\[
f(x, y) = \begin{cases} 
\frac{6x^2 - 5x^3y + 4y^2}{3x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\
2 & \text{if } (x, y) = (0, 0)
\end{cases}
\]
is differentiable at \((0, 0)\).

**Solution.** We have \( f(x, y) = 2 + g(x, y) \), where \( g(x, y) = \frac{-5x^2y^2}{3x^2 + 2y^2} \) for \((x, y) \neq (0, 0)\), and \( g(0, 0) = 0 \).
We claim the linearization of \( f \) at \((0, 0)\) is \( L(x, y) = 2 \); to prove this, and hence show that \( f \) is differentiable at \((0, 0)\), we need to show that
\[
\lim_{(x, y) \to (0, 0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = 0,
\]
or in other words, that
\[
\lim_{(x, y) \to (0, 0)} \frac{g(x, y)}{\sqrt{x^2 + y^2}} = 0.
\]
Here is the proof:
Given \( \varepsilon > 0 \), let \( \delta = \varepsilon / 5 > 0 \). Then for any \((x, y)\) with \( 0 < \sqrt{x^2 + y^2} < \delta \), note that \( 3x^2 + 2y^2 \geq x^2 + y^2 \), and that \( 5x^2y^2 \leq 5(x^2 + y^2)^2 \). Thus,
\[
\left| \frac{g(x, y)}{\sqrt{x^2 + y^2}} - 0 \right| = \frac{5x^2y^2}{(3x^2 + 2y^2)\sqrt{x^2 + y^2}} \leq \frac{5(x^2 + y^2)^2}{(x^2 + y^2)^{3/2}} = 5\sqrt{x^2 + y^2} < 5\delta = \varepsilon.
\]
QED

**OPTIONAL BONUS B. (1 point)** Name the five countries that are permanent members of the United Nations Security Council.

**Answer.** USA, Russia, France, UK, China. (I.e., the victors in World War II.)