Math 211, Section 05, Fall 2018

Solutions to Midterm Exam 1

1. (15 points) Find an equation for the line of intersection of the planes \( x - y + 2z = 1 \) and \( x + 3z = 2 \)

**Solution.** The normal vectors to the planes are \( \vec{n}_1 = (1, -1, 2) \) and \( \vec{n}_2 = (1, 0, 3) \). So to get a vector \( \vec{v} \) parallel to the desired line, we take their cross product:

\[
\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = (-3 - 0)\vec{i} + (2 - 3)\vec{j} + (0 + 1)\vec{k} = (-3, -1, 1).
\]

To find a point on the line, we [arbitrarily] set \( z = 0 \) and solve the two equations \( x - y = 1 \) and \( x = 2 \). That quickly gives \( x = 2 \) and \( y = 1 \), so the point \( (2, 1, 0) \) is on the line. Thus, \( \vec{r}(t) = (-3t + 2, -t + 1, t) \) is a parametric equation for the line.

2. (15 points) Let \( \vec{a} = (1, 4, -2) \) and \( \vec{b} = (2, -1, 3) \).

   (2a) Let \( \theta \) be the angle between \( \vec{a} \) and \( \vec{b} \). Compute either \( \cos \theta \) or \( \sin \theta \).
   (Your choice, but specify which one you are computing).

   (2b) Compute the (vector) projection \( \text{proj}_{\vec{b}} \vec{a} \) of \( \vec{a} \) onto \( \vec{b} \).

**Solution.** (a): We have \( \vec{a} \cdot \vec{b} = 2 - 4 - 6 = -8 \) and \( \|\vec{a}\| = \sqrt{1 + 16 + 4} = \sqrt{21} \), and

\[
\|\vec{b}\| = \sqrt{4 + 1 + 9} = \sqrt{14}. \quad \text{So } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-8}{\sqrt{21}\sqrt{14}} = \frac{-8}{7\sqrt{6}}.
\]

(b): \( \text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = \frac{-8}{14} (2, -1, 3) = \left[\frac{-4}{7} (2, -1, 3)\right] \)

3. (15 points) Let \( L_1 \) be the line given by \( \vec{r}_1(t) = \langle 2t + 1, 1 - t, 3 + t \rangle \), and let \( L_2 \) be the line given by \( \vec{r}_2(t) = \langle t - 2, 1 - 2t, 1 \rangle \).

   (3a) Show that \( L_1 \) and \( L_2 \) intersect, and find their point of intersection.

   (3b) Find an equation for the plane perpendicular to \( L_1 \) that passes through the point \( (2, -5, 6) \).

**Solution.** (a): Solving \( \vec{r}_1(s) = \vec{r}_2(t) \) gives \( 2s + 1 = t - 2, 1 - s = 1 - 2t, 3 + s = 1 \).

The last equation gives \( s = -2 \), so the first gives \( t = 2(-2) + 1 + 2 = -1 \). We confirm this works:

\[
\vec{r}_1(-2) = (-3, 3, 1) = \vec{r}_2(-1).
\]

So the two lines intersect at \( (-3, 3, 1) \).

(b): The normal vector to the plane must be parallel to \( L_1 \), so we use

\[
\vec{n} = \vec{v}_1 = (2, -1, 1)
\]

For a point on the plane, we use \( (2, -5, 6) \). So the plane has equation

\[
2(x - 2) - 1(y + 5) + 1(z - 6) = 0, \quad \text{i.e.,} \quad 2x - y + z = 15
\]

4. (30 points) Sketch the graphs of the following two quadric surfaces. Provide either some “trace” curves in separate graphs, or some brief verbal description, or both, to help explain what your graph is trying to show, and to explain how you obtained it.

   (4a) (15 points) \( x^2 - 4y^2 - z^2 = 0 \) \quad (4b) (15 points) \( x^2 - 4y^2 + z^2 = -9 \)

**Solution.** (a): The \( yz \)-plane trace \( (x = 0) \) is \( 4y^2 + z^2 = 0 \), which is a single point at the origin.

The \( xz \)-plane trace \( (y = 0) \) is \( x^2 = z^2 \), which is two crossing lines.

The \( xy \)-plane trace \( (z = 0) \) is \( x^2 = 4y^2 \), which is two crossing lines.

Since the \( x = 0 \) trace was degenerate, we try \( x = k \), which is \( 4y^2 + z^2 = k^2 \), which is an ellipse.
Putting all the traces together, the surface is an elliptical double cone [not that you need to know the name] opening along the $x$-axis. See me for a graph.

(b): The $yz$-plane trace ($x = 0$) is $4y^2 - z^2 = 9$, which is a hyperbola opening along the $y$-axis.

The $xz$-plane trace ($y = 0$) is $x^2 + z^2 = -9$, which is no points at all.

The $xy$-plane trace ($z = 0$) is $-x^2 + 4y^2 = 9$, which is a hyperbola opening along the $y$-axis.

Since the $y = 0$ trace was degenerate, we try $y = k$, which is $x^2 + z^2 = 4k^2 - 9$. For $|k| < 3/2$, that’s nothing at all; but for $|k| > 3/2$, that’s a circle.

Putting all the traces together, the surface is a hyperboloid of two sheets [not that you need to know the name] opening along the $y$-axis. (That is, one bowl opening to the right, and one to the left, when viewed from our usual viewpoint.) See me for a graph.

5. **(10 points)** Write out a definite integral whose value is the length of the curve given by $\vec{r}(t) = \langle t^2, \sqrt{t+3}, \cos(\pi t) \rangle$ from the point $(4, 1, 1)$ to the point $(1, 2, -1)$.

**Do not evaluate the integral.** As always, don’t forget to justify your answer.

**Solution.** We have $\vec{r}'(t) = \left\langle 2t, \frac{1}{2}(t + 3)^{-1}/2(1), -\pi \sin(\pi t) \right\rangle$, so $\|\vec{r}'(t)\| = \sqrt{4t^2 + \frac{1}{4(t + 3)} + \pi^2 \sin^2(\pi t)}$.

The point $(4, 1, 1)$ is $\vec{r}(-2)$ (which we can see mainly by looking at the $y$-coordinate), and the point $(1, 2, -1)$ is $\vec{r}(1)$. So the integral is $\int_{-2}^{1} \sqrt{4t^2 + \frac{1}{4(t + 3)} + \pi^2 \sin^2(\pi t)} \, dt$.

6. **(15 points)** Let $\vec{r}(t) = \langle \sin t, 3t, \cos t \rangle$.

(a) Compute $\vec{r}(0)$ and $\vec{r}'(0)$.

(b) Graph the curve traced out by $\vec{r}(t)$. On your graph, mark the point $\vec{r}(0)$, with the vector $\vec{r}'(0)$ emanating from it.

**Solution.** (a). $\vec{r}(0) = \langle 0, 0, 1 \rangle$, and $\vec{r}'(t) = \langle \cos t, 3, -\sin t \rangle$, so $\vec{r}''(0) = \langle 1, 3, 0 \rangle$.

(b). The curve satisfies $x^2 + z^2 = 1$, so it lies in the circular cylinder given by that equation, which is centered along the $y$-axis. Moreover, as $t$ increases, the motion rotates clockwise around the circle when viewed from the left. Also as $t$ increases, $y$ increases. So the curve is a helix curling around the $y$-axis, moving from left to right, and curling clockwise when viewed from the left. See me for a picture; it’s important to get $\vec{r}'(0)$ actually tangent to the curve at $(0, 0, 1)$.

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**OPTIONAL BONUS A. (2 points)** Let $\vec{r}(t)$ be a vector-valued function (specifically, taking values in $\mathbb{R}^3$ whose third derivative $\vec{r}'''$ is defined. Prove that

$$\frac{d}{dt} \left[ \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) \right] = \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}'''(t)).$$

**Proof.** Using the product rules for $\cdot$ and $\times$, as well as properties of the triple product:

$$\frac{d}{dt} \left[ \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) \right] = \vec{r}'(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) + \vec{r}(t) \cdot (\vec{r}''(t) \times \vec{r}'(t) + \vec{r}'(t) \times \vec{r}'''(t)) = (\vec{r}'(t) \times \vec{r}'(t)) \cdot \vec{r}'''(t) + \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}'''(t)) = 0 + \vec{r}(t) \cdot (\vec{r}''(t) \times \vec{r}'''(t)) = \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}'''(t)).$$

[Note: Another way to see that $\vec{r}'(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) = 0$ is to notice that by properties of $\times$, the cross product $\vec{r}'(t) \times \vec{r}''(t)$ is perpendicular to $\vec{r}'(t)$, and therefore its dot product with $\vec{r}'(t)$ is 0.]

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**OPTIONAL BONUS B. (1 point)** Name the current heads of state of two of the following three countries: Hungary, Syria, and the Philippines.

**Answers.** Hungary: Victor Orbán. (I’d also accept János Áder, the president, although that is a less powerful role in Hungary than Prime Minister, Orbán’s position.) Syria: Bashar al-Assad. Philippines: Rodrigo Duterte.