Final Exam, Thursday, December 19, 2019

Instructions: Do all twelve numbered problems. If you wish, you may also attempt the three optional bonus questions. Show all work, including scratch work. Little or no credit may be awarded, even when your answer is correct, if you fail to follow instructions for a problem or fail to justify your answer. If your answer for a given problem is a sum of fractions with different denominators, you may leave it that way. Otherwise, simplify your answers whenever possible. If you need more space, use the back of any page. If you have time, check your answers.

WRITE LEGIBLY. NO CALCULATORS.

1. (12 points) Let \( f(x, y) = \ln(2x + y) \).
   (a) Write an equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((-1, 3, 0)\).
   (b) Use your tangent plane equation from part (a) to estimate \( f(-1.1, 2.9) \).

2. (18 points) Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function \( f(x, y) = x^2y - 2y^2 - 4y \).

3. (12 points) Find an equation for the plane that contains the point \((0, 3, 0)\) and the line \( \vec{r}(t) = \langle 4 - t, 1 + 2t, 3t \rangle \).

4. (18 points) Let \( f(x, y) = \begin{cases} \frac{x^3 + xy - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \)
   (a) Prove that \( f \) is not continuous at \((0, 0)\).
   (b) Compute the directional derivative \( D_\vec{u}f(0, 0) \), where \( \vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \).

5. (20 points) Find the absolute maximum and absolute minimum values of the function \( f(x, y) = xy^2 - 2x^3 \) on the circle \( x^2 + y^2 = 9 \).

6. (20 points) Let \( E \) be the solid lying inside the sphere \( x^2 + y^2 + z^2 = 4 \) and above the cone \( z = \sqrt{x^2 + y^2} \) in the first octant. Compute \( \iiint_E x \, dV \).

7. (15 points) Let \( C \) be the straight line segment from \((1, 0, 0)\) to \((0, -3, 2)\). Compute \( \int_C \vec{F} \cdot d\vec{r} \), where \( \vec{F}(x, y, z) = (2y, -xz, z^2) \).
8. (18 points) Let $C$ be the curve in the plane consisting of the straight line segment from $(0,0)$ to $(1,1)$, then around the arc of the circle $x^2 + y^2 = 2$ counterclockwise to the point $(-\sqrt{2},0)$, and finally straight along the $x$-axis back to $(0,0)$, as in this picture:

![Diagram of curve](image)

Compute $\int_C (x^2 + y^2) \, dx + 5xy \, dy$.

9. (12 points) Let $\vec{F}(x,y,z) = \langle 4xy + \cos x, \sin x - 2y^2, z \sin x \rangle$.

(a) Compute $\text{div} \, \vec{F}$ and $\text{curl} \, \vec{F}$.

(b) Decide whether $\vec{F}$ is the gradient of something, or the curl of something, or both, or neither. (You do not need to find the things it may be gradient or curl of, but you do need to (briefly) explain your reasoning.)

10. (15 points) Let $\vec{F}(x,y) = \langle 2xy + 6x^2, x^2 - y^3 \rangle$.

(a) Show that $\vec{F}$ is conservative by finding a potential function $f(x,y)$ for $\vec{F}$.

(b) Let $C$ be the curve parametrized by $\vec{r}(t) = \langle t^2 - 2, 4t - 2t^2 \rangle$, for $1 \leq t \leq 2$.

Compute $\int_C \vec{F} \cdot d\vec{r}$.

11. (20 points) Let $S$ be the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$, oriented outward.

Let $\vec{G}(x,y,z) = \langle 2x^2 - yz, x^4, y^3 - 2xz \rangle$. Compute the flux $\int_S \vec{G} \cdot d\vec{S}$ of $\vec{G}$ through $S$.

12. (20 points) Let $S$ be the portion of the elliptic paraboloid $y = x^2 + z^2$ to the left of the plane $y = 1$, oriented with normal vectors pointing to the left (i.e., towards the negative-$y$ direction).

Let $\vec{F}(x,y,z) = \langle x, 0, x \rangle$. Compute the flux $\int_S \vec{F} \cdot d\vec{S}$ of $\vec{F}$ through $S$.

OPTIONAL BONUS A. (2 points) Consider the circle $C$ in the $yz$-plane given by the equation $(y - 3)^2 + z^2 = 1$. Let $S$ be the surface formed by revolving $C$ around the $z$-axis. (Note: $S$ is called a torus.) Find a parametrization of the surface $S$.

OPTIONAL BONUS B. (2 points) Let $\vec{F} = \langle y^2 - xz, x^3 - yz, xy + z^2 \rangle$. Find a vector field $\vec{G}$ such that $\text{curl} \, \vec{G} = \vec{F}$.

OPTIONAL BONUS C. (1 point) Earlier this week, a court in Pakistan convicted a former president of that nation of treason and sentenced him to death. Name this former president of Pakistan.