Some Tips on Writing ε-δ Proofs

Suppose you need to prove that \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) by ε-δ. Your written proof should have scratchwork (where you strategize about how to choose δ) clearly set aside, followed by the formal proof. Only the formal proof is really the answer, but any student learning this for the first time (or even tenth time) should really write out the scratchwork. The full writeup, including scratchwork and proof, should have the following format:

\[ \text{Scratch:} \]
(Some informal doodling/strategizing.)

\[ \vdots \]
(End of Scratchwork.)

\[ \text{Proof.} \]
Given \( \varepsilon > 0 \), let \( \delta = \boxed{\text{[Formula for \delta]}}, \) > 0.

Given \( (x, y) \) such that \( 0 < \| (x, y) - (a, b) \| < \delta \),

\[ \vdots \]
So \( |f(x,y) - L| < \varepsilon \). QED

(Of course, the box above should be filled in with whatever formula you ended up choosing for \( \delta \).) Note that the expression \( \| (x, y) - (a, b) \| \) is just \( \sqrt{(x-a)^2 + (y-b)^2} \), i.e., the distance from \( (x, y) \) to \( (a, b) \).

The point of the scratchwork is to figure out what goes in the box for \( \delta \), and to figure out how to get from \( 0 < \| (x, y) - (a, b) \| < \delta \) to \( |f(x,y) - L| < \varepsilon \). Generally the best way to proceed is to go backwards. That is, start from

\[ |f(x,y) - L| < \varepsilon \]

and try to change it (via algebra and inequalities) to turn it into something involving only \( \sqrt{(x-a)^2 + (y-b)^2} \). Then figure out how small you’d want \( \sqrt{(x-a)^2 + (y-b)^2} \) to be (i.e., how small \( \delta \) should be) to get the whole thing less than \( \varepsilon \).

Some useful equalities and inequalities:

\[ |AB| = |A||B|, \]
\[ \frac{|A|}{|B|} = \frac{|A|}{|B|}, \]
\[ |A + B| \leq |A| + |B| \]
\[ |A| \leq \sqrt{A^2 + B^2}, \]
\[ |B| \leq \sqrt{A^2 + B^2}. \]
Please note a few features of the format on the previous page:

1. The scratchwork is clearly identified as just scratchwork, and it is physically separated from the rest of the writeup by brackets or lines or something.

2. The proof begins
   “Proof. Given \( \varepsilon > 0 \). Let \( \delta = \cdots \). Given \((x, y)\) such that \(0 < \| (x, y) - (a, b) \| < \varepsilon\).”
   These four items (a clear start to the proof, the identification of \( \varepsilon \), the choice of \( \delta \), and then the identification of \( x \)) must be there, and they must be in that order.

3. The proof ends with the sentence \( |f(x, y) - L| < \varepsilon \), or perhaps a longer sentence of the form \( |f(x, y) - L| = \cdots < \varepsilon \).

4. Finally, make it clear that the proof is complete by writing “QED” or a box (\( \square \)).

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**Example.** Prove \( \lim_{(x,y)\to(0,0)} \frac{3x^2 - xy}{\sqrt{x^2 + y^2}} = 0 \).

**SCR:**

\[
\left| \frac{3x^2 - xy}{\sqrt{x^2 + y^2}} - 0 \right| = \frac{|3x^2 - xy|}{\sqrt{x^2 + y^2}} \leq \frac{|3x^2| + |xy|}{\sqrt{x^2 + y^2}} = \frac{3|x|^2 + |x||y|}{\sqrt{x^2 + y^2}} \\
\leq \frac{3(\sqrt{x^2 + y^2})^2 + (\sqrt{x^2 + y^2})^2}{\sqrt{x^2 + y^2}} = 4\sqrt{x^2 + y^2} < 4\delta
\]

Want \( \varepsilon \), so let \( \delta = \varepsilon/4 \).

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**Proof.** Given \( \varepsilon > 0 \), let \( \delta = \varepsilon/4 > 0 \).

Given \((x, y)\) such that \(0 < \sqrt{x^2 + y^2} < \delta\), we have

\[
\left| \frac{3x^2 - xy}{\sqrt{x^2 + y^2}} - 0 \right| = \frac{|3x^2 - xy|}{\sqrt{x^2 + y^2}} \leq \frac{|3x^2| + |xy|}{\sqrt{x^2 + y^2}} = \frac{3|x|^2 + |x||y|}{\sqrt{x^2 + y^2}} \\
\leq \frac{3(\sqrt{x^2 + y^2})^2 + (\sqrt{x^2 + y^2})^2}{\sqrt{x^2 + y^2}} = 4\sqrt{x^2 + y^2} < 4\delta = \varepsilon.
\]

QED

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