Geometric Series Test (GST)

Consider a series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$. This Geometric series

$$\begin{cases} \text{Converges if } |r| < 1, \quad \text{with SUM} = \dfrac{a}{1-r} \\ \\ \text{Diverges if } \ |r| \geq 1 \end{cases}$$

USED: For series where each successive term is found by multiplying the previous term by a common ratio r. These series contain terms that look "exponential-ish" where there is a fixed base raised to changing or variable powers.

NOTE: Do not worry if the given power is not exactly of the form $n - 1$. You do **not** need to muscle your series terms to match the $n - 1$ form. In fact that usually leads to more mistakes.

THINK: If the Common Ratio Multiplier r is *tiny*, smaller than 1, then the terms are shrinking fast enough for the Series to Converge. Otherwise, if the Multipler r is not *small* then the terms are growing, and then the series won't have a chance to converge.

APPROACH:

- Write out at least the first three terms for a geometric series. The first term is always a, and the multiplying factor to compute each successive term is the common ratio r. Technically you only need the first two terms to find a and r, but it is strongly recommended to write a third term so you can confirm that you have indeed chosen the correct Multiplier r.
- Determine if $|r|$ is less than 1 or greater than or equal to 1. The test says you must check the **absolute value** of r. Even if r is positive, it is recommended to write $|r|$ because that is the condition in the convergence test. Finally compare $|r|$ to 1, and make a clear declaration of convergence or divergence by the GST.
- If $|r| \geq 1$, so that the series diverges, then you are done. If $|r| < 1$, so that the series converges, then you can compute the actual sum of the full original geometric series. Here the SUM=$\frac{a}{1-r}$. Please simplify.

NOTE: Pay attention to whether the Geometric Series starts with index counter $n = 0$ or say $n = 1$ since that can change the first term value a. A different starting counter will not change the Multiplier r Common Ratio value.

KEY: Pay attention to the sign(s) of a and r, since that can change your final Sum for convergent cases.

EXAMPLES: Determine and state whether each of the following series **Converges** or **Diverges**. Name any Convergence test(s) that you use, and justify all of your work. If the Geometric series converges, compute the SUM.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3^{n+2}}{2^{3n-1}} = -\frac{3^3}{2^2} + \frac{3^4}{2^5} - \frac{3^5}{2^8} + \dots$

Here we have a geometric series with $a = -\frac{27}{4}$ and $r = -\frac{3}{2^3} = -\frac{3}{8}$. It $\boxed{\text{Converges by GST}}$ since $|r| = \left|-\frac{3}{8}\right| = \frac{3}{8} < 1$.

As a result, the sum is given by SUM$=\dfrac{a}{1-r} = \dfrac{-\frac{27}{4}}{1-\left(-\frac{3}{8}\right)} = \dfrac{-\frac{27}{4}}{\frac{11}{8}} = -\frac{27}{4}\cdot\frac{8}{11} = \boxed{-\frac{54}{11}}$

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{4^{3n-1}}{7^{n+3}} = -\frac{4^2}{7^4} + \frac{4^5}{7^5} - \frac{4^8}{7^6} + \dots$

Here we have a geometric series with $r = -\frac{4^3}{7} = -\frac{64}{7}$. It $\boxed{\text{Diverges by GST}}$ since $|r| = \left|-\frac{64}{7}\right| = \frac{64}{7} > 1$.

(Note you don't need a here because the series diverges and you will NOT be computing the sum.)

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{4^{2n+1}}{3^{4n-1}} = -\frac{4^3}{3^3} + \frac{4^5}{3^7} - \frac{4^7}{3^{11}} + \dots$

Here we have a geometric series with $a = -\frac{64}{27}$ and $r = -\frac{4^2}{3^4} = -\frac{16}{81}$. It $\boxed{\text{Converges by GST}}$ since $|r| = \left|-\frac{16}{81}\right| = \frac{16}{81} < 1$.

As a result, the sum is given by SUM$=\dfrac{a}{1-r} = \dfrac{-\frac{64}{27}}{1-\left(-\frac{16}{81}\right)} = \dfrac{-\frac{64}{27}}{\frac{97}{81}} = -\frac{64}{27}\cdot\frac{81}{97} = \boxed{-\frac{192}{97}}$