Extra Examples of Interval and Radius of Convergence

Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-4)^n}{5^n\,\sqrt{n}}$ Use Ratio Test. $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}(3x-4)^{n+1}}{\sqrt{n+1}\;5^{n+1}}}{\dfrac{\cancel{(-1)^{n}}(3x-4)^{n}}{\sqrt{n}\;5^{n}}}\right| = \lim_{n\to\infty}\left|\frac{(3x-4)^{n+1}}{(3x-4)^n}\right|\cdot\left(\sqrt{\frac{\cancel{n}^{\,\to 1}}{\cancel{n+1}^{\,\to 1}}}\right)^{\to 1}\cdot\frac{5^n}{5^{n+1}} = \frac{|3x-4|}{5}$$

The Ratio Test gives convergence for x when $\dfrac{|3x-4|}{5} < 1$ or $|3x-4| < 5$.

(Note: Must make this statement above)

That is, $-5 < 3x-4 < 5 \Longrightarrow -1 < 3x < 9 \Longrightarrow -\dfrac{1}{3} < x < 3$

Manually Test Endpoints: (where $L = 1$ and Ratio Test is Inconclusive)

•$x = 3$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(3(3)-4)^n}{\sqrt{n+1}\;5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\cancel{5^n}}{\sqrt{n+1}\;\cancel{5^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

which is Convergent by AST: 1. $b_n = \dfrac{1}{\sqrt{n+1}} > 0$ 2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{\cancel{n+1}^{\,\to\infty}}} = 0$

3. Terms decreasing: $b_{n+1} = \dfrac{1}{\sqrt{n+2}} < \dfrac{1}{\sqrt{n+1}} = b_n \Rightarrow\ x = 2$ is Included in the Domain.

•$x = -\dfrac{1}{3}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\left(3\left(-\frac{1}{3}\right)-4\right)^n}{\sqrt{n+1}\;5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-5)^n}{\sqrt{n+1}\;5^n}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\cancel{5^n}}{\sqrt{n+1}\;\cancel{5^n}} = \sum_{n=1}^{\infty}\frac{\cancel{(-1)^{2n^{\text{even}}}}^{\,\to 1}}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}} \approx \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$ the Divergent p-Series $p = \frac{1}{2}$.

LCT: $\displaystyle\lim_{n\to\infty}\frac{\dfrac{1}{\sqrt{n+1}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{\cancel{n}^{\,\to 1}}{\cancel{n+1}^{\,\to 1}}} = 1$ which is *Finite* and *Non-zero*.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$ is also Divergent by LCT $\Rightarrow\ x = -\dfrac{1}{3}$ is NOT included in the Domain.

Finally, Interval of Convergence $\boxed{I = \left(-\dfrac{1}{3}, 3\right]}$ with Radius of Convergence $\boxed{R = \dfrac{5}{3}}$.

Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(b) $\sum_{n=1}^{\infty} n!\,(x-6)^n$ Use Ratio Test. $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!(x-6)^{n+1}}{n!(x-6)^n} \right| \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} |x-6| = \lim_{n\to\infty} \cancel{(n+1)}^{\infty} |x-6| = \infty > 1$$

Diverges by the Ratio Test for all x *unless* $x - 6 = 0$ or $x = 6$ (when $L = 0 < 1$)

(Note: Must make this statement above)

So $\boxed{I = \{6\}}$ with $\boxed{R = 0}$.

(c) $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ Use Ratio Test. $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(2(n+1))!}}{\frac{x^n}{(2n)!}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} |x| \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \lim_{n\to\infty} \frac{|x|}{\cancel{(2n+2)(2n+1)}_{\to\infty}} = 0 < 1$$

for all x

Converges by the Ratio Test for all Real numbers x

(Note: Must make this statement above)

$\boxed{I = (-\infty, \infty)}$ with $\boxed{R = \infty}$.