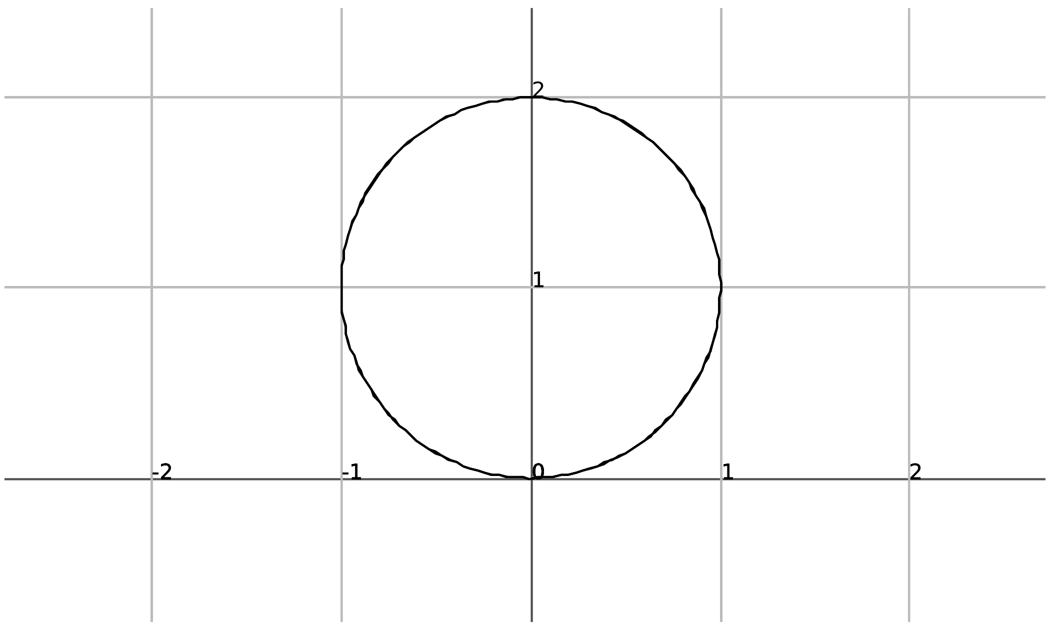
Review Packet for Final Exam

Material since Exam #3

Math 121-D. Benedetto

Polar Coordinates: For each problem, sketch the polar curve(s) and answer the related question(s).

1. Find the area bounded by $r = 2\sin\theta$.



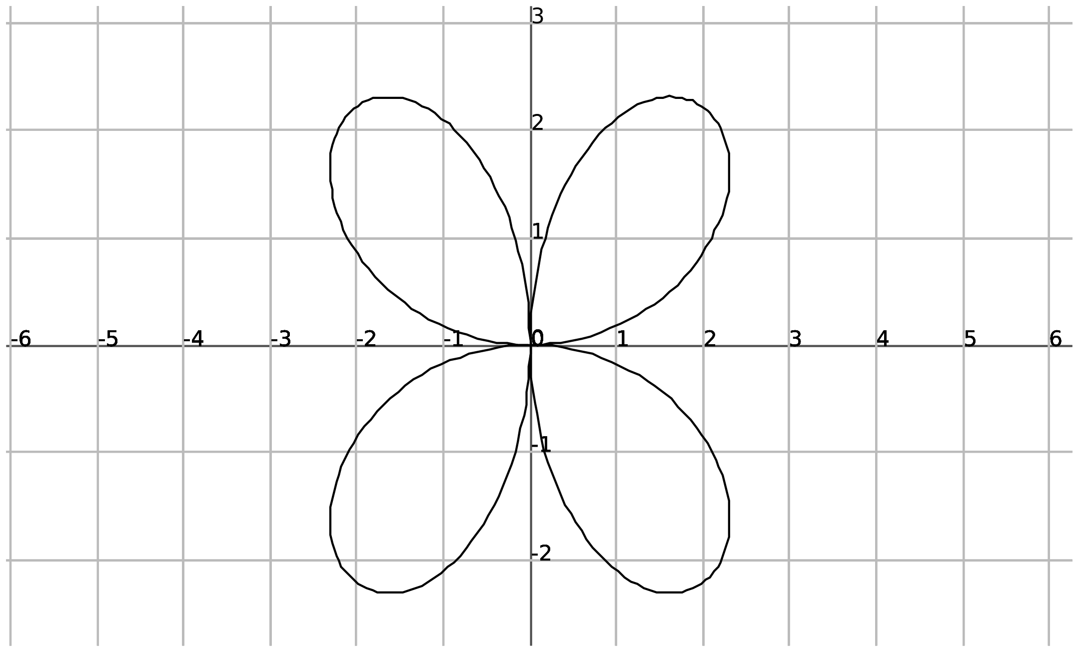
First note that this is just a circle of radius 1, so the area should be π. We will use the area formula for polar curves to double check this. It's important to note that one cycle of the polar circle closes up on itself as θ ranges from $\theta = 0$ to just $\theta = \pi$.

$$\text{Area} = A = \frac{1}{2}\int_0^{\pi} r^2\ d\theta = \frac{1}{2}\int_0^{\pi} (2\sin\theta)^2\ d\theta = \frac{1}{2}\int_0^{\pi} 4\sin^2\theta\ d\theta = 2\int_0^{\pi} \sin^2\theta\ d\theta$$

$$= 2\int_0^{\pi} \frac{1-\cos(2\theta)}{2}\ d\theta = \int_0^{\pi} 1-\cos(2\theta)\ d\theta = \theta - \frac{\sin(2\theta)}{2}\Big|_0^{\pi} = \left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)$$

$$= \boxed{\pi}$$

2. Find the area bounded by one petal loop of the 4 leaved rose $r = 3\sin(2\theta)$.



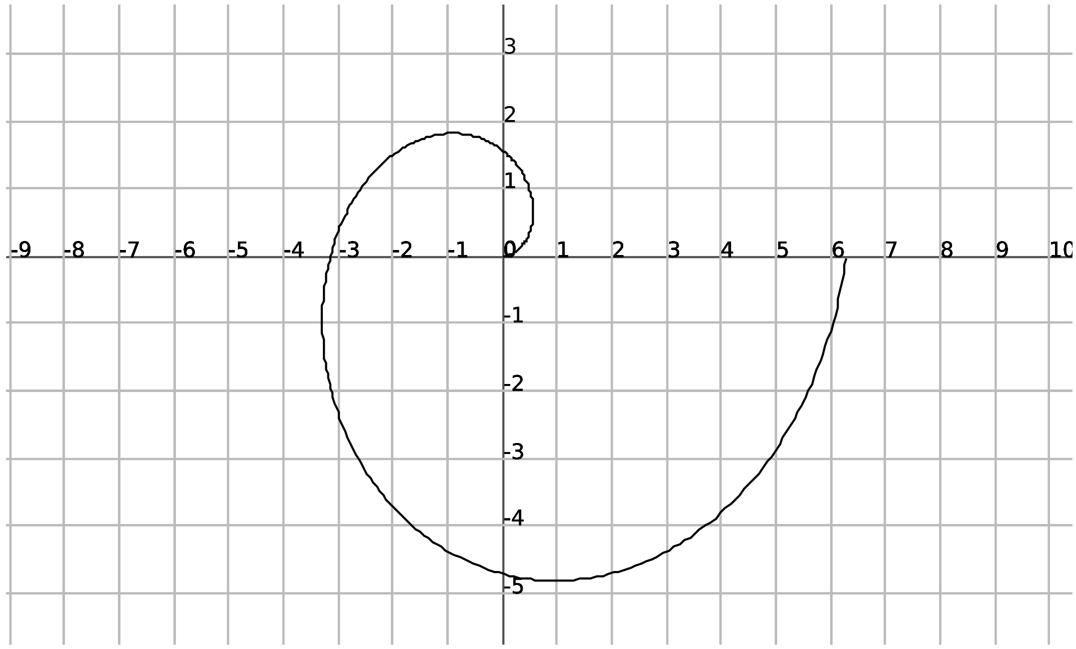
It's important to note that one loop of the polar petal closes up on itself as θ ranges from $\theta = 0$ to just $\theta = \frac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (3\sin(2\theta))^2\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta)\ d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\ d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\ d\theta = \frac{9}{4}\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\ d\theta = \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

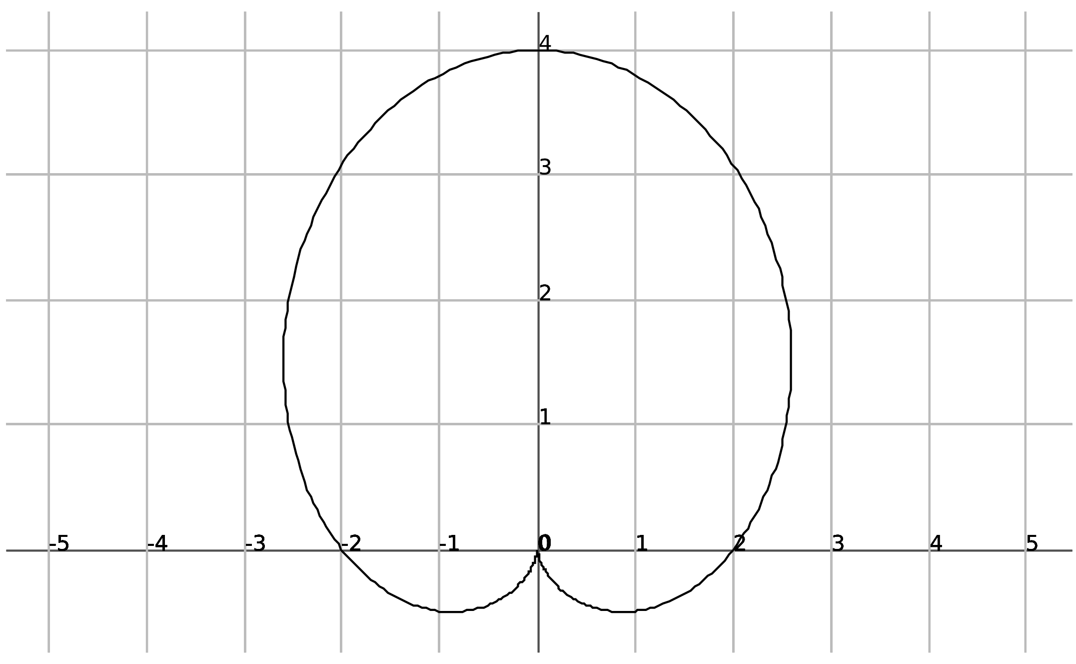
$$= \frac{9}{4}\left(\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(0 - \frac{\sin 0}{2}\right)\right) = \boxed{\frac{9\pi}{8}}$$

3. Find the area bounded by $r = \theta$ with $0 \leq \theta \leq 2\pi$



$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{1}{2}\left(\frac{\theta^3}{3}\right)\Big|_0^{2\pi} = \frac{1}{2}\left(\frac{(2\pi)^3}{3} - 0\right) = \frac{1}{2}\left(\frac{8\pi^3}{3}\right) = \boxed{\frac{4\pi^3}{3}}$$

4. Find the area bounded by the cardioid $r = 2 + 2\sin\theta$.



It's important to note that one full cycle of the cardioid closes up on itself as θ ranges from $\theta = 0$ to $\theta = 2\pi$.

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (2 + 2\sin(\theta))^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta$$

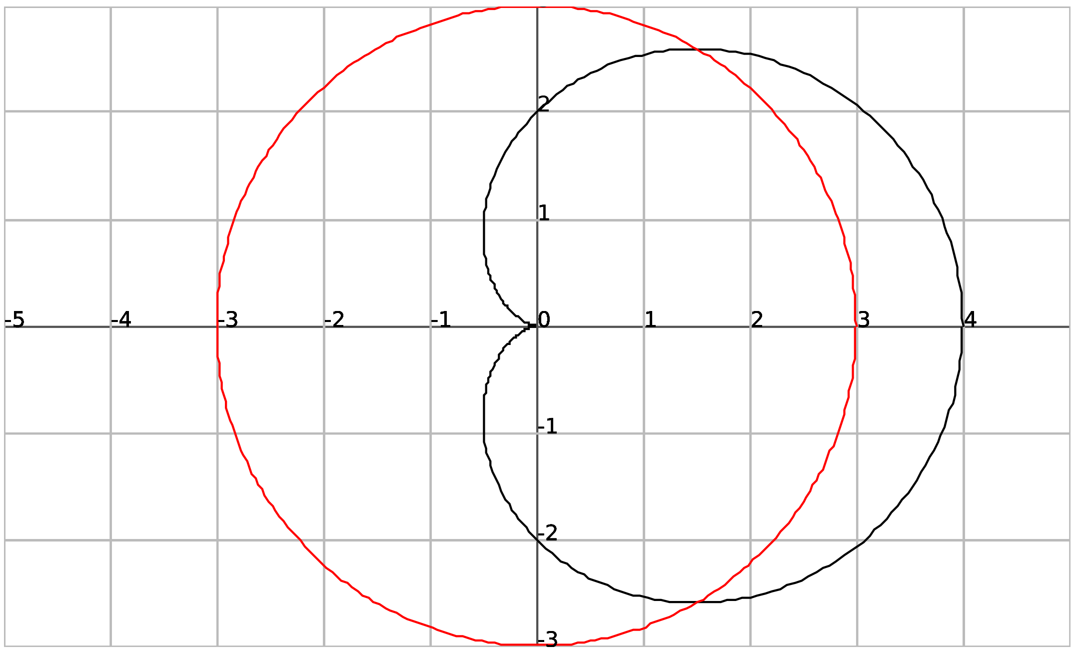
$$= \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 4\left(\frac{1-\cos(2\theta)}{2}\right)\ d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 2\left(1-\cos(2\theta)\right)\ d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 6 + 8\sin\theta - 2\cos(2\theta)\ d\theta = \frac{1}{2}\left(6\theta - 8\cos\theta - \sin(2\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left((12\pi - 8\cos(2\pi) - \sin(4\pi)) - (0 - 8\cos 0 - \sin 0)\right)$$

$$\frac{1}{2}\left((12\pi - 8 - 0) - (0 - 8 - 0)\right) = \frac{1}{2}(12\pi - 8 + 8) = \boxed{6\pi}$$

5. Find the area bounded inside $r = 2 + 2\cos\theta$ and outside $r = 3$.



These two polar curves intersect when $2 + 2\cos\theta = 3 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right) = \int_0^{\frac{\pi}{3}} \left((2+2\cos\theta)^2 - 3^2\right)\ d\theta$$

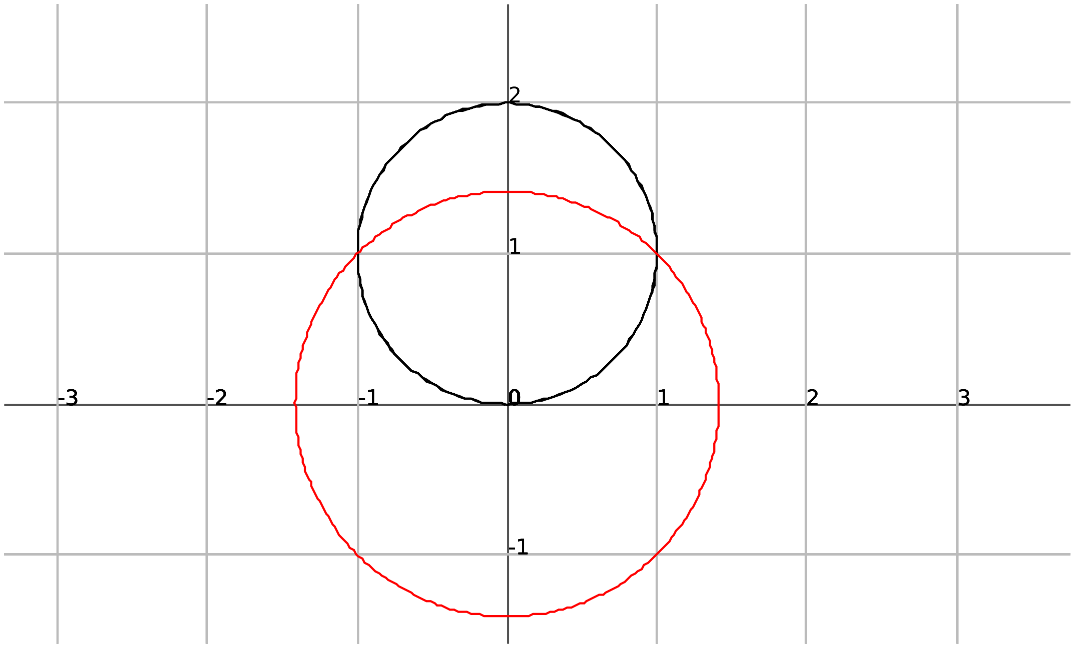
$$= \int_0^{\frac{\pi}{3}} 4 + 8\cos\theta + 4\cos^2\theta - 9\ d\theta = \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 4\left(\frac{1+\cos(2\theta)}{2}\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 2 + 2\cos(2\theta)\ d\theta = \int_0^{\frac{\pi}{3}} -3 + 8\cos\theta + 2\cos(2\theta)\ d\theta$$

$$= -3\theta + 8\sin\theta + \sin(2\theta)\Big|_0^{\frac{\pi}{3}} = \left(-\pi + 8\sin\left(\tfrac{\pi}{3}\right) + \sin\left(\tfrac{2\pi}{3}\right)\right) - (0 + 8\sin 0 + \sin 0)$$

$$= -\pi + 8\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right) = -\pi + 4\sqrt{3} + \tfrac{\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{2} - \pi}$$

6. Find the area bounded inside $r = 2\sin\theta$ and outside $r = \sqrt{2}$.



These two polar curves intersect when $2\sin\theta = \sqrt{2} \Rightarrow \sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$. Using symmetry, we will integrate from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ and double that area.

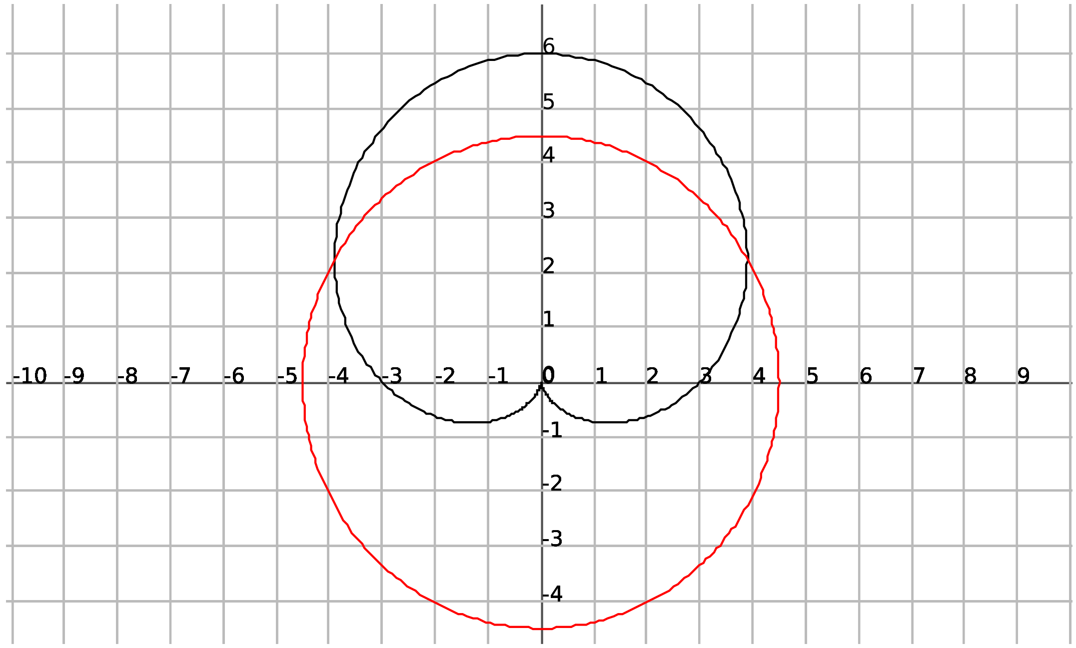
$$\text{Area} = A = 2\left(\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left((2\sin\theta)^2 - (\sqrt{2})^2\right)\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\sin^2\theta - 2\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\left(\frac{1-\cos(2\theta)}{2}\right) - 2\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2 - 2\cos(2\theta) - 2\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -2\cos(2\theta)\ d\theta$$

$$= -\sin(2\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\sin\pi - \left(-\sin\left(\tfrac{\pi}{2}\right)\right) = 0 + 1 = \boxed{1}$$

7. Find the area bounded inside $r = 3 + 3\sin\theta$ and outside $r = \frac{9}{2}$.



These two polar curves intersect when $3 + 3\sin\theta = \frac{9}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ to $\theta = \frac{5\pi}{6}$. Using symmetry, we will integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((\text{outer } \ r)^2 - (\text{inner } \ r)^2\right)\, d\theta\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((3 + 3\sin\theta)^2 - \left(\frac{9}{2}\right)^2\right)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + 9\sin^2\theta - \frac{81}{4}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + 9\left(\frac{1-\cos(2\theta)}{2}\right) - \frac{81}{4}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + \frac{9}{2} - \frac{9}{2}\cos(2\theta) - \frac{81}{4}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\frac{27}{4} + 18\sin\theta - \frac{9}{2}\cos(2\theta)\, d\theta$$

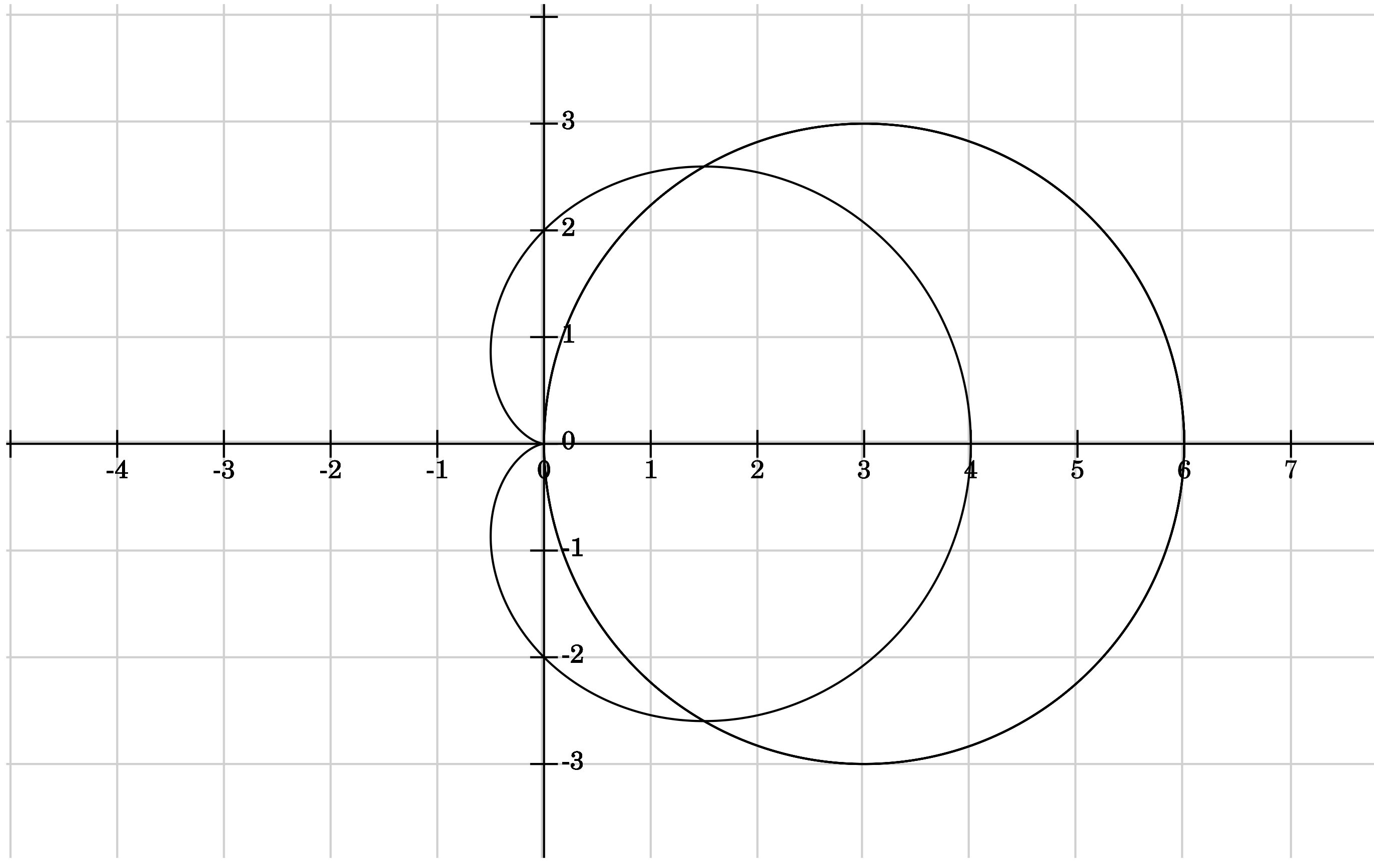
$$= -\frac{27}{4}\theta - 18\cos\theta - \frac{9}{4}\sin(2\theta)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(-\frac{27\pi}{8} - 18\cos\left(\frac{\pi}{2}\right) - \frac{9}{4}\sin(\pi)\right) - \left(-\frac{27\pi}{24} - 18\cos\left(\frac{\pi}{6}\right) - \frac{9}{4}\sin(\frac{\pi}{3})\right)$$

$$= -\frac{27\pi}{8} - 0 - 0 + \frac{9\pi}{8} + 18\left(\frac{\sqrt{3}}{2}\right) + \frac{9}{4}\left(\frac{\sqrt{3}}{2}\right) = -\frac{27\pi}{8} - 0 - 0 + \frac{9\pi}{8} + 9\sqrt{3} + \frac{9\sqrt{3}}{8}$$

$$= \frac{81\sqrt{3}}{8} - \frac{18\pi}{8} = \boxed{\frac{81\sqrt{3}}{8} - \frac{9\pi}{4}}$$

8. Find the area bounded outside the polar curve $r = 2 + 2\cos\theta$ and inside the polar curve $r = 6\cos\theta$.



These two polar curves intersect when $2+2\cos\theta = 6\cos\theta \Rightarrow 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((\text{outer } \ r)^2 - (\text{inner } \ r)^2\right)\ d\theta\right) = \int_0^{\frac{\pi}{3}} \left((6\cos\theta)^2 - (2+2\cos\theta)^2\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} 36\cos^2\theta - (4+8\cos\theta+4\cos^2\theta)\ d\theta = \int_0^{\frac{\pi}{3}} 36\cos^2\theta - 4 - 8\cos\theta - 4\cos^2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} 32\cos^2\theta - 4 - 8\cos\theta\ d\theta = \int_0^{\frac{\pi}{3}} 32\left(\frac{1+\cos(2\theta)}{2}\right) - 4 - 8\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} 16\left(1+\cos(2\theta)\right) - 4 - 8\cos\theta\ d\theta = \int_0^{\frac{\pi}{3}} 16 + 16\cos(2\theta) - 4 - 8\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} 12 + 16\cos(2\theta) - 8\cos\theta\ d\theta$$

$$= 12\theta + 8\sin(2\theta) - 8\sin\theta\Big|_0^{\frac{\pi}{3}} = \left(12\left(\frac{\pi}{3}\right) + 8\sin(\frac{2\pi}{3}) - 8\sin\left(\frac{\pi}{3}\right)\right) - (0+0-0)$$

$$= 4\pi + 8\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right) = \boxed{4\pi}$$