Math 121-03, Fall 2024 Professor Rob Benedetto

Solutions to the Final Exam

1. **(15 points)** Evaluate the following limits. Please **justify** your answers, using appropriate notation. Be clear about whether each limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0^+} \big(1-\sin x\big)^{1/x}$ **(b) Use series** to evaluate $\displaystyle\lim_{x\to 0} \frac{2x-\arctan(2x)}{(\sin x)-x}$

Solution. (a): $\displaystyle\lim_{x\to 0^+} \big(1-\sin x\big)^{1/x} \stackrel{1^\infty}{=} \exp\left(\lim_{x\to 0^+} \frac{\ln(1-\sin x)}{x}\right)$

$$\stackrel{(\frac{0}{0})^{\text{L'H}}}{=} \exp\left(\lim_{x\to 0^+} \frac{\frac{1}{1-\sin x}\cdot(-\cos x)}{1}\right) = \exp\left(\frac{1}{1-0}\cdot(-1)\right) = \exp(-1) = \boxed{\frac{1}{e}}$$

(b): $$\lim_{x\to 0} \frac{2x-\arctan(2x)}{(\sin x)-x} = \lim_{x\to 0} \frac{2x-\big(2x-\frac{(2x)^3}{3}+\frac{(2x)^5}{5}-\cdots\big)}{\big(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\big)-x}$$

$$= \lim_{x\to 0} \frac{\frac{8}{3}x^3-\frac{2^5}{5}x^5+\cdots}{-\frac{1}{6}x^3+\frac{1}{5!}x^5+\cdots} = \lim_{x\to 0} \frac{\frac{8}{3}-\frac{2^5}{5}x^2+\cdots}{-\frac{1}{6}+\frac{1}{5!}x^2+\cdots} = -\frac{8/3}{1/6} = -\frac{16}{1} = \boxed{-16}$$

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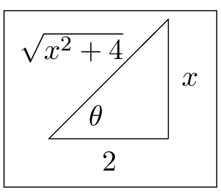
2. **(20 points)** Compute the following integrals by any legal method. Simplify your answers.

(a) $\displaystyle\int \frac{x^2}{(x^2+4)^{5/2}}\,dx$ **(b)** $\displaystyle\int_e^{e^3} x^2\ln x\,dx$

Solution. (a): $\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \\ x^2+4 = 2\sec^2\theta \end{array}}$ $\displaystyle = \int \frac{4\tan^2\theta}{(2^2\sec^2\theta)^{5/2}}\cdot 2\sec^2\theta\,d\theta = \int \frac{8\tan^2\theta\sec^2\theta}{2^5\sec^5\theta}\,d\theta$

$\displaystyle = \frac{1}{4}\int \sin^2\theta\cos\theta\,d\theta$ $\boxed{\begin{array}{l} u=\sin\theta \\ du = \cos\theta\,d\theta \end{array}}$ $\displaystyle = \frac{1}{4}\int u^2\,du = \frac{1}{12}u^3 + C$

$\displaystyle = \frac{1}{12}\sin^3\theta + C$



$\displaystyle = \boxed{\frac{x^3}{12(x^2+4)^{3/2}} + C}$

(b): $\boxed{\begin{array}{ll} u = \ln x & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{3}x^3 \end{array}}$ $\displaystyle = \frac{1}{3}x^3\ln x\Big|_e^{e^3} - \frac{1}{3}\int_e^{e^3} x^2\,dx = \left(\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right)\Big|_e^{e^3}$

$$= \left(\frac{1}{3}e^9\cdot 3 - \frac{1}{9}e^9\right) - \left(\frac{1}{3}e^3\cdot 1 - \frac{1}{9}e^3\right) = \boxed{\frac{8}{9}e^9 - \frac{2}{9}e^3}$$

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3. **(20 points)** For each of the following **Improper** integrals, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_5^\infty \frac{4}{x^2-4x+7}\,dx$ **(b)** $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx$

Solution. (a): $\displaystyle\int_5^\infty \frac{4}{x^2-4x+7}\,dx = \lim_{t\to\infty}\int_5^t \frac{4}{(x-2)^2+3}\,dx$ $\boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}}$

$$= \lim_{t\to\infty}\int_3^{t-2}\frac{4}{u^2+3}\,du = \lim_{t\to\infty}\frac{4}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t-2} = \lim_{t\to\infty}\left[\frac{4}{\sqrt{3}}\left(\arctan\left(\frac{t-2}{\sqrt{3}}\right)-\arctan\left(\sqrt{3}\right)\right)\right]$$

$$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{4}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{2\pi}{3\sqrt{3}}} \qquad \text{(Converges)}$$

(b): $\displaystyle\int_{-4}^{-3}\frac{8-x}{x^2+2x-8}\,dx = \lim_{t\to -4^+}\int_t^{-3}\frac{8-x}{(x+4)(x-2)}\,dx$

PFD: $\displaystyle\frac{8-x}{(x+4)(x-2)} = \frac{A}{x+4}+\frac{B}{x-2} = \frac{A(x-2)+B(x+4)}{(x+4)(x-2)} = \frac{(A+B)x+(-2A+4B)}{(x+4)(x-2)}$,

so $A+B=-1$ and $-2A+4B=8$.

Adding 2 times the first to the second equation gives $2B+4B = 2(-1)+8$,

so $6B=6$, and so $B=1$. Then $A=-1-B=-2$.

So we have $\displaystyle\frac{8-x}{(x+4)(x-2)} = \frac{-2}{x+4}+\frac{1}{x-2}$.

So the original integral is $\displaystyle\lim_{t\to -4^+}\int_t^{-3}\frac{-2}{x+4}+\frac{1}{x-2}\,dx = \lim_{t\to -4^+}\big(-2\ln|x+4|+\ln|x-2|\big)\Big|_t^{-3}$

$$= \lim_{t\to -4^+}\Big[\big(-2\ln 1+\ln 5\big)-\big(-2\ln(t+4)+\ln(2-t)\big)\Big] = 0+\ln 5+2\ln(0^+)-\ln 6 = \boxed{-\infty} \text{ (Diverges)}$$

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4. **(15 points)** Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and of course justify all of your work.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^3+1}$ **(b)** $\displaystyle\sum_{n=2}^{\infty}\frac{n^3+1}{\ln n}$

Solution. (a): We have $\displaystyle 0 \le \frac{\ln n}{n^3+1} \le \frac{n}{n^3} = \frac{1}{n^2}$.

We know $\displaystyle\sum\frac{1}{n^2}$ converges by the p-Test ($p=2>1$).

So OS $\boxed{\text{converges}}$ by CT.

(b): We have $\displaystyle\lim_{n\to\infty}\frac{n^3+1}{\ln n} = \lim_{x\to\infty}\frac{x^3+1}{\ln x} \overset{(\frac{\infty}{\infty})^{\text{L'H}}}{=} \lim_{x\to\infty}\frac{3x^2}{1/x} = \lim_{x\to\infty}3x^3 = \infty \ne 0$.

So OS $\boxed{\text{diverges}}$ by nTDT.

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5. **(20 points)** Determine whether each of the following series is **Absolutely Convergent**, **Conditionally Convergent**, or **Diverges**. Name any convergence test(s) you use, and of course justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3+5}{n^5+3}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+5}$

Solution. (a): AS: $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+5}{n^5+3}$

LCT: $\lim_{n\to\infty} \dfrac{\dfrac{n^3+5}{n^5+3}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \dfrac{n^5+5n^2}{n^5+3} = \lim_{n\to\infty} \dfrac{1+5n^{-3}}{1+3n^{-5}} = \dfrac{1}{1} = 1$ is > 0 and finite.

Now $\sum \dfrac{1}{n^2}$ converges by the p-Test ($p = 2 > 1$).
So AS converges by LCT.
So OS is $\boxed{\text{Absolutely Convergent}}$ by definition.

(b): Apply AST:

- $\dfrac{1}{3n+5} > 0$,
- $\lim_{n\to\infty} \dfrac{1}{3n+5} = \dfrac{1}{\infty} = 0$,
- $\dfrac{1}{3(n+1)+5} < \dfrac{1}{3n+5}$,

So OS converges by AST.

AS is $\sum_{n=1}^{\infty} \dfrac{1}{3n+5}$.

LCT: $\lim_{n\to\infty} \dfrac{\dfrac{1}{3n+5}}{\dfrac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{3n+5} = \lim_{n\to\infty} \dfrac{1}{3+5n^{-1}} = \dfrac{1}{3}$ is > 0 and finite.

Now $\sum \dfrac{1}{n}$ diverges by the p-Test ($p = 1$).
So AS diverges by LCT.
So OS is $\boxed{\text{Conditionally Convergent}}$ by definition.

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6. **(10 points)** Show that the MacLaurin series for $\dfrac{1}{2}(e^x + e^{-x})$ is $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$

Solution, Method 1. We have $e^x = \sum_{n=0}^{\infty} \dfrac{1}{n!}x^n$ and $e^{-x} = \sum_{n=0}^{\infty} \dfrac{1}{n!}(-x)^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!}x^n$.

So $\dfrac{1}{2}(e^x + e^{-x}) = \sum_{n=0}^{\infty} \dfrac{1}{2}\left(\dfrac{1}{n!} + \dfrac{(-1)^n}{n!}\right)x^n$.

For $n = 2m$ even, we have $\dfrac{1}{2}\left(\dfrac{1}{(2m)!} + \dfrac{(-1)^(2m)}{(2m)!}\right) = \dfrac{1}{2}\left(\dfrac{1}{(2m)!} + \dfrac{1}{(2m)!}\right) = \dfrac{1}{(2m)!}$

For $n = 2m+1$ odd, we have $\dfrac{1}{2}\left(\dfrac{1}{(2m+1)!} + \dfrac{(-1)^(2m+1)}{(2m+1)!}\right) = \dfrac{1}{2}\left(\dfrac{1}{(2m+1)!} - \dfrac{1}{(2m+1)!}\right) = 0$

Thus, $\dfrac{1}{2}(e^x + e^{-x}) = \sum_{n=0}^{\infty} \dfrac{1}{2}\left(\dfrac{1}{n!} + \dfrac{(-1)^n}{n!}\right)x^n = \sum_{m=0}^{\infty} \dfrac{1}{(2m)!}x^{2m} = \sum_{n=0}^{\infty} \dfrac{1}{(2n)!}x^{2n}$, as desired,
where the last equality is because it doesn't matter whether we name the index m or n.

Solution, Method 2. Chart method with $f(x) = \dfrac{1}{2}(e^x + e^{-x})$:

$n=0$	$f(x) = \frac{1}{2}(e^x + e^{-x})$	$f(0) = \frac{1}{2}(e^0 + e^0) = 1$
$n=1$	$f'(x) = \frac{1}{2}(e^x - e^{-x})$	$f'(0) = \frac{1}{2}(e^0 - e^0) = 0$
$n=2$	$f''(x) = \frac{1}{2}(e^x + e^{-x})$	$f''(0) = \frac{1}{2}(e^0 + e^0) = 1$
$n=3$	$f'''(x) = \frac{1}{2}(e^x - e^{-x})$	$f'''(0) = \frac{1}{2}(e^0 - e^0) = 0$
$\vdots$	$\vdots$	$\vdots$

So $f(x) = \frac{1}{0!} + \frac{0}{1!}x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \frac{1}{0!} + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

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7. **(25 points)** Find the **Interval of Convergence** and the **Radius of Convergence** for each of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x+1)^n}{n^3 \cdot 5^n}$ **(b)** $\sum_{n=1}^{\infty} n^n\ (x-5)^n$

Solution. (a): Ratio Test: $L = \lim_{n\to\infty} \left| \frac{\frac{(3x+1)^{n+1}}{(n+1)^3 \cdot 5^{n+1}}}{\frac{(3x+1)^n}{n^3 \cdot 5^n}} \right| = \lim_{n\to\infty} \left| \frac{(3x+1)^{n+1}}{(3x+1)^n} \right| \cdot \frac{n^3}{(n+1)^3} \cdot \frac{5^n}{5^{n+1}}$

$= \lim_{n\to\infty} |3x+1| \cdot \frac{1}{(1+\frac{1}{n})^3} \cdot \frac{1}{5} = \frac{1}{5}|3x+1|$

We have $L < 1$ for $-1 < \frac{1}{5}(3x+1) < 1$, i.e. $-5 < 3x+1 < 5$, i.e. $-6 < 3x < 4$, i.e. $-2 < x < \frac{4}{3}$.

Endpoint $x = -2$: the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{n^3 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which is a p-series with $p = 3 > 1$ and so converges by the p-Test.

Endpoint $x = 4/3$: the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(5)^n}{n^3 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$.

The corresponding absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which we just saw converge; so our current series also converges by ACT.

[Alternatively, one could show convergence by checking the three conditions of AST.]

So the $\boxed{\text{interval of convergence is } \left[-2, \frac{4}{3}\right]}$, which has length $\frac{4}{3} - (-2) = \frac{10}{3}$.

So the $\boxed{\text{radius of convergence is } \frac{5}{3}}$ since that is half the length.

(a): Ratio Test: $L = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}(x-5)^{n+1}}{n^n(x-5)^n} \right| = \lim_{n\to\infty} \cancel{\left(\frac{n+1}{n}\right)^{n}} \to e \cdot (n+1) \cdot |x-5|$,
which is $0 < 1$ if $x = 5$, and otherwise is $\infty > 1$.

so the $\boxed{\text{interval of convergence is } \{5\}}$ (single point), and the $\boxed{\text{radius of convergence is } 0}$

8. **(30 points)** Each of the following series converges. For each one, **find its sum**. Simplify.

(a) $\frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \cdots$ **(b)** $\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{3!\ 2^n\ n!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ \pi^{2n}}{(36)^n\ (2n+1)!}$ **(d)** $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

(e) $1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \cdots$ **(f)** Show that $\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \boxed{-\frac{35}{18}}$

Solution. (a): $= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) - 2\left(1 - \frac{1}{2}\right) = 2\big(\ln(1+1)\big) - 2 + 1 = \boxed{2\ln 2 - 1}$

(b): $= \frac{1}{3!} \sum_{n=0}^{\infty} \frac{(-1)^n\ (\frac{\ln 9}{2})^n}{n!} = \frac{1}{6} e^{\ln(9)/2} = \frac{1}{6} e^{\ln 3} = \frac{3}{6} = \boxed{\frac{1}{2}}$

(c): $= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ (\frac{\pi}{6})^{2n}}{(2n+1)!} = -\frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\ (\frac{\pi}{6})^{2n+1}}{(2n+1)!} = -\frac{6}{\pi} \sin\left(\frac{\pi}{6}\right) = -\frac{6}{\pi} \cdot \frac{1}{2} = \boxed{-\frac{3}{\pi}}$

(d): $= -\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = -\arctan(1) = \boxed{-\frac{\pi}{4}}$

(e): $= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = 1 + \cos(\pi) = 1 - 1 = \boxed{0}$

(f): $\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \sum_{n=0}^{\infty} \left(\frac{-4}{5}\right)^n - \sum_{n=0}^{\infty} \frac{2}{5^n}$

Both of these two sums are geometric. The first has ratio $r_1 = -\frac{4}{5}$, and the second has $r_2 = \frac{1}{5}$. Since $|r_1|, |r_2| < 1$, both sums converge, and the original series converges to

$$\frac{1}{1 - \left(-\frac{4}{5}\right)} - \frac{2}{1 - \frac{1}{5}} = \frac{5}{5+4} + \frac{10}{5-1} = \frac{5}{9} - \frac{5}{2} = \frac{5(2-9)}{18} = \boxed{-\frac{35}{18}}$$

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9. **(15 points)** Find the MacLaurin series representation for $\ln(9 + x^2)$.

[*Suggestion*: You may use the formula $\ln(9 + x^2) = \int \frac{2x}{9 + x^2}\,dx$. Don't forget to solve for C.]

Solution, Method 1. We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $\frac{1}{9 - 9x} = \frac{1}{9} \cdot \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{x^n}{9}$.

Substitute $-\frac{x^2}{9}$ for x: $\frac{1}{9 + x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^{n+1}}$.

Multiply by $2x$: $\frac{2x}{9 + x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2x^{2n+1}}{9^{n+1}}$.

Antidifferentiate: $\ln(9 + x^2) = C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2x^{2n+2}}{(2n+2) \cdot 9^{n+1}} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1) \cdot 9^{n+1}}$

To find C, plug in $x = 0$: $\ln(9) = C + 0$, so $C = \ln 9$.

So the MacLaurin series is $\boxed{\ln(9+x^2) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1)\cdot 9^{n+1}}}$ or alternatively, $\ln 9 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n \cdot 9^n}$

or alternatively, $\ln 9 + \frac{1}{9}x^2 - \frac{1}{2\cdot 9^2}x^4 + \frac{1}{3\cdot 9^3}x^6 - \frac{1}{4\cdot 9^4}x^8 + \cdots$

Solution, Method 2. We have $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$,

so $\ln(9+9x) = \ln 9 + \ln(1+x) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.

Substitute $\frac{x^2}{9}$ for x: $\boxed{\ln(9+x^2) = \ln 9 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1)\cdot 9^{n+1}}}$

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10. **(10 points)** Use Series to Estimate $\int_0^1 x^3 \sin\left(x^2\right)\, dx$ with error less than $\frac{1}{10,000}$.

You may leave your answer as a sum or difference of a few fractions.

[*Free Tips*: $(120)\cdot(14) = 1680$ **and** $7! = 5040$ **and** $(5040)\cdot(18) = 90,720$]

Solution. Substituting x^2 in the power series for sine:

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$, so $x^3\sin(x^2) = x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{17}}{7!} + \cdots$. Thus:

$$\int_0^1 x^3 \sin\left(x^2\right)\, dx = \frac{x^6}{6} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{14}}{14\cdot 5!} - \frac{x^{18}}{18\cdot 7!} + \cdots \Big|_0^1 = \frac{1}{6} - \frac{1}{10\cdot 3!} + \frac{1}{14\cdot 5!} - \frac{1}{18\cdot 7!} + \cdots$$

This is an alternating series [and one can check that the absolute values of the terms decrease to zero], and the fourth term has absolute value $\frac{1}{18\cdot 7!} = \frac{1}{90,720} < \frac{1}{10,000}$. [Using one of the free tips.]

Thus, by ASET, the first three terms give an accurate enough estimate, of

$\frac{1}{6} - \frac{1}{10\cdot 6} + \frac{1}{14\cdot 120} = \boxed{\frac{1}{6} - \frac{1}{60} + \frac{1}{1680}}$

(which is $\frac{280-28+1}{1680} = \frac{253}{1680}$, but we could leave it as the above sum of fractions).

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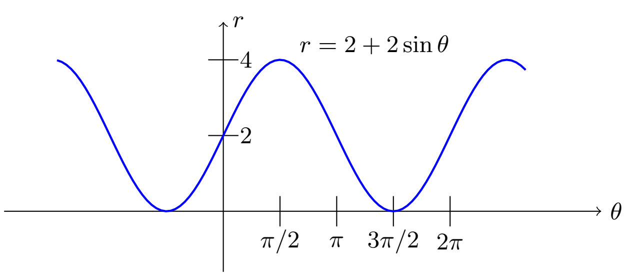
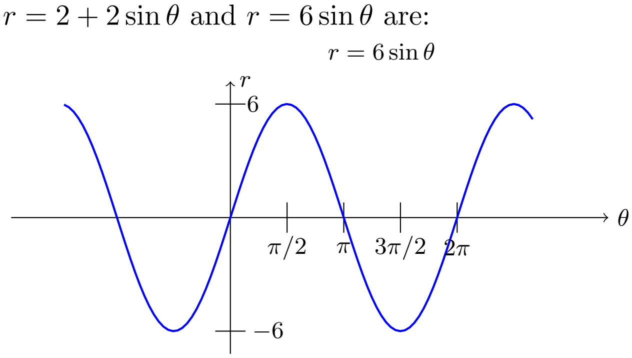
11. **(10 points)** Sketch the region that lies **outside** the polar curve $r = 2 + 2\sin\theta$ and **inside** the polar curve $r = 6\sin\theta$.

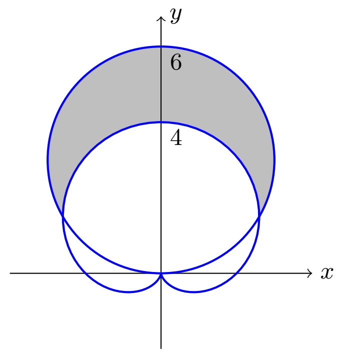
Then set up (but do not compute) an integral giving the area of this region.

Solution. The rectangular-coordinates graphs of $r = 2 + 2\sin\theta$ and $r = 6\sin\theta$ are:





So here is the (polar coordinates) picture:



The two curves intersect where $2 + 2\sin\theta = 6\sin\theta$, i.e., $4\sin\theta = 2$, i.e., $\sin\theta = \dfrac{1}{2}$. This happens for $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$ [which is consistent with what the picture shows]. The picture also shows that the outer edge of the region is the circle $r = 6\theta$, and the inner edge is the cardioid.

So the area is $\boxed{\displaystyle\int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[(6\sin\theta)^2 - (2+2\sin\theta)^2\right] d\theta}$

Alternatively, using symmetry, one could instead write

$$2\int_{\pi/6}^{\pi/2} \frac{1}{2}\left[(6\sin\theta)^2 - (2+2\sin\theta)^2\right] d\theta = \int_{\pi/6}^{\pi/2} (6\sin\theta)^2 - (2+2\sin\theta)^2\, d\theta$$

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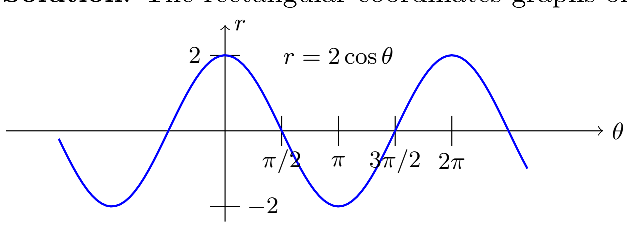
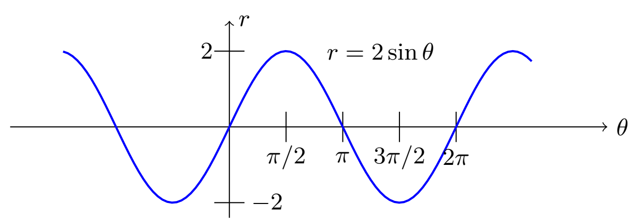
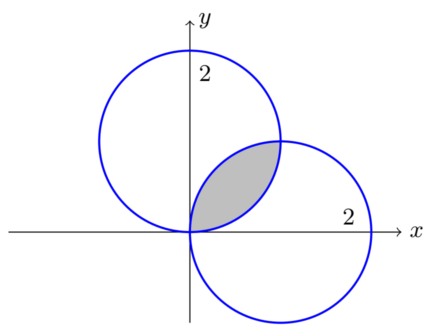
12. **(10 points)** Sketch the region that lies **inside** both of the polar curves $r = 2\cos\theta$ and $r = 2\sin\theta$. Then set up **AND compute** an integral giving the area of this region.

Solution. The rectangular-coordinates graphs of $r = 2\cos\theta$ and $r = 2\sin\theta$ are:

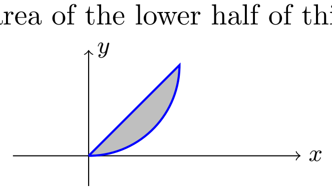




So here is the (polar coordinates) picture:



By symmetry, the area is twice the area of the lower half of this region:



The slanted line segment is at angle $\pi/4$ (from the origin to the other point where the two circles cross, where $2\sin\theta = 2\cos\theta$, i.e., for $\theta = \pi/4$). So the area of the original region is

$$2\int_0^{\pi/4} \frac{1}{2}\big(2\sin\theta\big)^2\, d\theta = 4\int_0^{\pi/4} \sin^2\theta\, d\theta = 2\int_0^{\pi/4} 1-\cos 2\theta\, d\theta = 2\theta - \sin 2\theta\Big|_0^{\pi/4}$$

$$= \left(\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right) - (0-\sin 0) = \boxed{\frac{\pi}{2}-1}$$

OPTIONAL BONUS A. (2 points.) Compute $\displaystyle\int \frac{16e^{3x}}{e^{4x}-16}\,dx$.

Solution. Substitute: $\boxed{\begin{array}{l} u = e^x \\ du = e^x\,dx \end{array}}$

$$\int \frac{16e^{3x}}{e^{4x}-16}\,dx = \int \frac{16u^2}{u^4-16}\,du = \int \frac{16u^2}{(u-2)(u+2)(u^2+4)}\,du$$

PFD: $\displaystyle\frac{16u^2}{(u-2)(u+2)(u^2+4)} = \frac{A}{u-2} + \frac{B}{u+2} + \frac{Cu+D}{u^2+4}$ gives

$16u^2 = (A+B+C)u^3 + (2A-2B+D)u^2 + (4A+4B-4C)u + (8A-8B-4D)$, so that

$$\begin{array}{rrrrcr} A & +B & +C & & = & 0 \\ 2A & -2B & & +D & = & 16 \\ A & +B & -C & & = & 0 \\ 2A & -2B & & -D & = & 0 \end{array}$$

Subtracting the first and third equations gives $2C = 0$, so that $C = 0$.
Subtracting the second and fourth equations gives $2D = 16$, so that $D = 8$.
The first equation now gives $A + B = 0$, and the second gives $2A - 2B = 8$, so that $A - B = 4$.
Adding these two equations gives $2A = 4$, so that $A = 2$, and hence $B = -2$.

That is, the PFD is $\displaystyle\frac{16u^2}{(u-2)(u+2)(u^2+4)} = \frac{2}{u-2} - \frac{2}{u+2} + \frac{8}{u^2+4}$.

So the original integral is $\displaystyle\int \frac{2}{u-2} - \frac{2}{u+2} + \frac{8}{u^2+4}\,du = 2\ln|u-2| - 2\ln|u+2| + 4\arctan\left(\frac{u}{2}\right) + C =$

$$\boxed{2\ln|e^x-2| - 2\ln|e^x+2| + 4\arctan\left(\frac{1}{2}e^x\right) + C}$$

OPTIONAL BONUS B. (2 points.) Find the **interval** of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n$.

Solution. Ratio Test: $\displaystyle L = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)!\,x^{n+1}}{(n+1)^{n+1}}}{\dfrac{n!\,x^n}{n^n}} \right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \cdot \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^n\cdot(n+1)}$

$$= \lim_{n\to\infty} |x| \cdot \cancel{(n+1)} \cdot \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to 1/e} \cdot \frac{1}{\cancel{n+1}} = \frac{|x|}{e}$$

So $L < 1$ gives $|x| < e$. That is, the series converges for $-e < x < e$ and diverges for $|x| > e$. We need to check the endpoints.

Endpoint $x = e$: The series is $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, e^n}{n^n}$.

Recall that $e^n = 1 + n + \dfrac{n^2}{2!} + \dfrac{n^3}{3!} + \cdots$. The n-th term of *this* series is $\dfrac{n^n}{n!}$, and all the rest of its terms are positive. So we must have $e^n \geq \dfrac{n^n}{n!}$. Thus, $\dfrac{n!\, e^n}{n^n} \geq 1$ for all integers $n \geq 1$.

In particular, $\displaystyle\lim_{n\to\infty} \frac{n!\, e^n}{n^n} \neq 0$. So the series $\displaystyle\sum \frac{n!\, e^n}{n^n}$ diverges by nTDT.

Endpoint $x = -e$: The series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n!\, e^n}{n^n}$. We just saw that the terms of this series do not approach zero. So the series $\displaystyle\sum \frac{(-1)^n\, n!\, e^n}{n^n}$ diverges by nTDT.

Thus, neither endpoint is in the interval of convergence. So the interval of convergence is $\boxed{(-e, e)}$

OPTIONAL BONUS C. (1 point.) Less than three weeks ago, the (democratically elected) president of a certain nation declared martial law, only to have the declaration reversed by the nation's legislature a few hours later. A few days ago, the legislature impeached the president. What is the nation, and what is the name of the (now impeached) president in question?

Answer. The nation is South Korea (or the Republic of Korea), and the president is Yoon. (Full name: Yoon Suk Yeol.)