Math 121 Final May 19, 2022

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [16 Points] (a) Use Series to show that $\displaystyle\lim_{x\to 0} \frac{5x^2+\arctan(3x)-3x}{2x-\sin(2x)-3x^2} = \boxed{-\frac{5}{3}}$

(b) Compute $\displaystyle\lim_{x\to 0} \frac{5x^2+\arctan(3x)-3x}{2x-\sin(2x)-3x^2}$ again using L'Hôpital's Rule.

2. [28 Points] Compute the following integrals. Simplify.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{7}{2}}}\,dx$ (b) $\displaystyle\int x\arctan x\,dx$ (c) $\displaystyle\int \sqrt{9-x^2}\,dx$

(d) Show that $\displaystyle\int_{\ln 2}^{\ln 5} \frac{2e^x}{e^{2x}-1}\,dx \overset{\text{hint}}{=} \int_{\ln 2}^{\ln 5} \frac{2\,e^x}{(e^x)^2-1}\,dx = \boxed{\ln 2}$

3. [30 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_{-1}^{6} \frac{15-x}{x^2-6x-7}\,dx$ Free Hint: $\displaystyle\frac{15-x}{(x-7)(x+1)} \overset{\text{PFD}}{=} \frac{1}{x-7}-\frac{2}{x+1}$

(b) $\displaystyle\int_{-\infty}^{6} \frac{1}{x^2-6x+12}\,dx$ (c) $\displaystyle\int_{0}^{e} x^2\cdot\ln x\,dx$ (d) $\displaystyle\int_{0}^{\frac{1}{2}} \frac{1}{x\ln x}\,dx$

4. [24 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\dots$ (b) $\frac{1}{e}-\frac{1}{2e^2}+\frac{1}{3e^3}-\frac{1}{4e^4}+\frac{1}{5e^5}-\frac{1}{6e^6}+\dots$

(c) $2-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\dots$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,\pi^{2n}}{9^n\,(2n+1)!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2^{n+1}\,(\ln 3)^n}{3!\cdot n!}$ (f) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n-1}{3^n} = \boxed{-\frac{9}{10}}$

5. [12 Points] Determine whether each series **Converges** or **Diverges**. Name any convergence test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(n+6)^6} + \frac{1}{6^n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^6}{\ln n}$

6. [16 Points] Determine whether each given series is **Absolutely Convergent** or **Conditionally Convergent**. Name any convergence test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+7}{n^7+3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

7. [24 Points] Find the **Interval** and **Radius** of Convergence for the Series in (a) and (b)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x+2)^n}{(n+7)^2\ 4^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} n^n\ (x-7)^n$

(c) Show that the MacLaurin Series for $\sin x$ has Infinite Radius of Convergence.

8. [12 Points] (a) Use Series to compute $\displaystyle\frac{d}{dx}\left(6x^3 \arctan(6x)\right)$ Answer in Sigma $\displaystyle\sum_{n=0}^{\infty}$ notation.

(b) Use Series to Estimate $\displaystyle\int_0^1 x^3 \cos\left(x^3\right)\ dx$ with error less than $\displaystyle\frac{1}{200}$

9. [14 Points] Answers in Sigma $\displaystyle\sum_{n=0}^{\infty}$ notation. **State** the Radius of Convergence for each.

(a) Find the MacLaurin Series Representation for $\displaystyle\frac{1}{(1+7x)^2}$. Hint: $\displaystyle\frac{1}{(1+7x)^2} = \frac{d}{dx}\left(\frac{-1}{7(1+7x)}\right)$

(b) Find the MacLaurin Series Representation for $\ln(3+x)$. Hint: $\displaystyle\ln(3+x) = \int \frac{1}{3+x}\, dx$.

Yes, solve for C

10. [24 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded outside the polar curve $r = 2 + 2\sin\theta$ and inside $r = 6\sin\theta$.

(b) The **Area** that lies inside both of the curves $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.

(c) The **Area** bounded inside both of the polar curves $r = 2\cos\theta$ and $r = 2\sin\theta$.

(d) The **Area** bounded outside $r = 1 - \cos\theta$ and inside $r = -3\cos\theta$.