Math 121 Final Exam December 19, 2019

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or ~~$\cosh(\ln 3)$~~ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [12 Points] Evaluate the following **limits**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $\displaystyle\lim_{x\to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x}$ (b) Compute $\displaystyle\lim_{x\to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x}$ **again** using series.

2. [18 Points] Evaluate the following **integral**.

(a) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sin^2 x\right)^{\frac{7}{2}}}\, dx = \frac{43}{60\sqrt{2}}$ (b) $\displaystyle\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\, dx$

3. [40 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_0^5 \frac{6}{x^2-4x-5}\, dx$ (b) $\displaystyle\int_0^{e^5} \frac{1}{x\left[25+(\ln x)^2\right]}\, dx$

(c) $\displaystyle\int_{-\infty}^5 \frac{6}{x^2-4x+7}\, dx$ (d) $\displaystyle\int_1^2 \frac{1}{x\ln x}\, dx$ (e) $\displaystyle\int_0^e \frac{\ln x}{\sqrt{x}}\, dx$

4. [18 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n - 2}{4^n}$ (Hint: split?) (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(\ln 9)^n}{2^{n+1}\cdot n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n-1}}{9^n\,(2n)!}$

(d) $\displaystyle\frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \frac{\pi^9}{9!} + \dots$ (e) $\displaystyle -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$ (f) $\displaystyle 3 - 1 + \frac{3}{5} - \frac{3}{7} + \frac{3}{9} - \dots$

5. [24 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+7}{n^7+2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{7} + \frac{7}{\arctan n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{n}+7}{n}\right)$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}$

6. [20 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-5)^n}{(n+7)^2 \cdot 7^{n+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} n!\,(x-6)^n$

7. [10 Points] Please analyze with detail and justify carefully. Simplify.

(a) Use MacLaurin Series to **Estimate** $\dfrac{1}{\sqrt{e}}$ with error less than $\dfrac{1}{100}$.

(b) Compute the MacLaurin Series for $f(x) = \dfrac{1}{(1-x)^2}$ and then **State** the Radius of Convergence. Your answer should be in Sigma notation. Hint: Use Differentiation.

8. [10 Points] For both parts, you do **not** need to find the Radius of Convergence. Your answer should be in Sigma notation or write out the first 5 non-zero terms.

(a) Demonstrate one method to compute the MacLaurin Series for $\boldsymbol{f(x) = \ln(1+x)}$. Justify. Do not just write down the formula.

(b) Demonstrate a second, **different** method to compute the MacLaurin Series for $\boldsymbol{f(x) = \ln(1+x)}$. Justify. Do not just write down a formula.

9. [10 Points]

(a) Write the **first 6 non-zero terms** of the MacLaurin Series for $\boldsymbol{f(x) = \sin\left(x^3\right) + \cos\left(x^3\right)}$.

~~(b) Use this series to now determine the **sixth, seventh, eighth** and **ninth** derivatives of $f(x) = \sin(x^3) + \cos(x^3)$ evaluated at $x = 0$. Do Not Simplify your answers.~~

10. [18 Points]

~~(a) Consider the Parametric Curve represented by $x = (\arctan t) + t$ and $y = 2\sinh^{-1} t$.~~

~~Recall $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$~~

~~**COMPUTE** the **arclength** of this parametric curve for $0 \le t \le \sqrt{3}$.~~

~~(b) Consider a different Parametric Curve represented by $x = \cos^3 t$ and $y = \sin^3 t$.~~

~~**COMPUTE** the **surface area** obtained by rotating this curve about the y-axis for $0 \le t \le \frac{\pi}{2}$.~~

11. [20 Points] For **each** of the following problems, do the following **THREE** things:

1. Sketch the Polar curve(s) and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 3 + 3\cos\theta$ and inside $r = 9\cos\theta$.

(b) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin\theta$.

(c) The **area** that lies inside both of the curves $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$.

~~(d) The **area** bounded inside **one** petal of the curve $r = 3\sin(2\theta)$.~~