Solutions to Midterm Exam 3

1. **(20 points)** Find the **Interval** and **Radius of Convergence** of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+4)^n}{n^2 \cdot 2^n}$

Solution. Ratio Test: $L = \displaystyle\lim_{n\to\infty} \left| \frac{\frac{(3x+4)^{n+1}}{(n+1)^2 \cdot 2^{n+1}}}{\frac{(3x+4)^n}{n^2 \cdot 2^n}} \right| = \lim_{n\to\infty} \left| \frac{(3x+4)^{n+1}}{(3x+4)^n} \right| \cdot \frac{n^2}{(n+1)^2} \cdot \frac{2^n}{2^{n+1}}$

$= \displaystyle\lim_{n\to\infty} |3x+4| \cdot \frac{1}{(1+\frac{1}{n})^2} \cdot \frac{1}{2} = \frac{1}{2}|3x+4|$

We have $L < 1$ for $-1 < \frac{1}{2}(3x+4) < 1$, i.e. $-2 < 3x+4 < 2$, i.e. $-6 < 3x < -2$, i.e. $-2 < x < -\frac{2}{3}$.

Endpoint $x = -2/3$: the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-2+4)^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$ and so converges by the p-Test.

Endpoint $x = -2$: the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-6+4)^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

The corresponding absolute series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which we just saw converge; so our current series also converges by ACT.

[Alternatively, one could show convergence by checking the three conditions of AST.]

So the $\boxed{\text{interval of convergence is } \left[-2, -\frac{2}{3}\right]}$, which has length $-\frac{2}{3} - (-2) = \frac{4}{3}$.

So the $\boxed{\text{radius of convergence is } \frac{2}{3}}$ since that is half the length.

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2. **(16 points)** For each of the following functions, find its MacLaurin series, written in Sigma notation $\displaystyle\sum_{n=0}^{\infty}$, **and also** find the **radius of convergence** of the series.

2a. $x^3 \arctan(5x^2)$ 2b. $\displaystyle\int x \sin\left(\frac{x^3}{2}\right) dx$

Solution. 2a. We know $\arctan x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$, for $|x| \le 1$.

So $\arctan(5x^2) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (5x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+2}$,

for $|5x^2| \le 1$, i.e., $5x^2 \le 1$, i.e., $x^2 \le \frac{1}{5}$, i.e., $|x| \le \frac{1}{\sqrt{5}}$.

So $x^3 \arctan(5x^2) = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+5}}$ still for $|x| \le \frac{1}{\sqrt{5}}$. So the $\boxed{\text{radius of convergence is } \frac{1}{\sqrt{5}}}$

2b. We know $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, for all x.

So $\sin\left(\frac{x^3}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}\cdot(2n+1)!}(x^3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}\cdot(2n+1)!} x^{6n+3}$

and hence $x\sin\left(\frac{x^3}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}\cdot(2n+1)!} x^{6n+4}$, still for all x.

So $\int x\sin\left(\frac{x^3}{2}\right)dx = \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(6n+5)\cdot(2n+1)!} x^{6n+5}}$ and the $\boxed{\text{radius of convergence is } \infty}$

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3. **(25 points, 6 parts, 3 pages)** Find the **sum** of each of the following convergent series. Simplify if possible.

3a. $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$ 3b. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{16^n(2n+1)!}$ 3c. $\sum_{n=0}^{\infty} \frac{2^n}{\pi^n \cdot n!}$

3d. $8 - \frac{8}{3} + \frac{8}{5} - \frac{8}{7} + \frac{8}{9} - \frac{8}{11} + \cdots$ 3e. $\sum_{n=0}^{\infty} \frac{(-1)^n(\ln 2)^n}{7\cdot(n!)}$ 3f. $\sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(-9)^n\cdot(2n)!}$

Solution. 3a. $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(1)^{n+1} = -\ln(1+1) = \boxed{-\ln 2}$

3b. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{16^n(2n+1)!} = \pi\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n}(2n+1)!} = 4\pi\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = 4\pi\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}$

$= 4\pi\sin\left(\frac{\pi}{4}\right) = 4\pi\cdot\frac{\sqrt{2}}{2} = \boxed{2\pi\sqrt{2}}$

3c. $\sum_{n=0}^{\infty} \frac{2^n}{\pi^n\cdot n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2}{\pi}\right)^n = \boxed{e^{2/\pi}}$

3d. $8 - \frac{8}{3} + \frac{8}{5} - \frac{8}{7} + \frac{8}{9} - \frac{8}{11} + \cdots = 8\sum_{n\geq 0} \frac{(-1)^n}{2n+1}(1)^{2n+1} = 8\arctan(1) = 8\cdot\frac{\pi}{4} = \boxed{2\pi}$

3e. $\sum_{n=0}^{\infty} \frac{(-1)^n(\ln 2)^n}{7\cdot(n!)} = \frac{1}{7}\sum_{n=0}^{\infty} \frac{1}{n!}(-\ln 2)^n = \frac{1}{7}e^{-\ln 2} = \frac{1}{7}\cdot\frac{1}{e^{\ln 2}} = \frac{1}{7}\cdot\frac{1}{2} = \boxed{\frac{1}{14}}$

3f. $\sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(-9)^n\cdot(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(2\pi)^{2n}}{3^{2n}\cdot(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(2\pi/3)^{2n}}{(2n)!} = \cos\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2}}$

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4. **(12 points)** Use MacLaurin series to estimate $\ln(1.1)$ with error less than 0.00005.

Solution. Writing $1.1 = 1 + \frac{1}{10}$, the MacLaurin series for $\ln(1+x)$ gives

$\ln\left(1.1\right) = \frac{1}{10} - \frac{1}{2}\cdot\frac{1}{100} + \frac{1}{3}\cdot\frac{1}{1000} - \frac{1}{4}\cdot\frac{1}{10,000} + \frac{1}{5}\cdot\frac{1}{100,000} - \cdots$

This is an alternating series [and one can check that the absolute values of the terms decrease to zero], and the fourth term has absolute value $\dfrac{1}{40,000} < \dfrac{1}{20,000} = 0.00005$.

Thus, by ASET, the first three terms give an accurate enough estimate, of $\boxed{\dfrac{1}{10} - \dfrac{1}{200} + \dfrac{1}{3000}}$ (which is $0.095\overline{3}$. The actual value is $\ln(1.1) = 0.0953101798\ldots$)

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5. **(12 points)** Use series to compute $\displaystyle\lim_{x\to 0} \frac{x^2 - \sin(x^2)}{\cos(x^3) - 1}$

Solution. $\displaystyle\lim_{x\to 0} \frac{x^2 - \sin(x^2)}{\cos(x^3) - 1} = \lim_{x\to 0} \frac{x^2 - \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right)}{\left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots\right) - 1} = \lim_{x\to 0} \frac{\frac{x^6}{6} - \frac{x^{10}}{5!} + \cdots}{-\frac{x^6}{2} + \frac{x^{12}}{4!} - \cdots}$

$$= \lim_{x\to 0} \frac{\frac{1}{6} - \frac{x^4}{5!} + \cdots}{-\frac{1}{2} + \frac{x^6}{4!} - \cdots} = \frac{1/6}{-1/2} = \boxed{-\frac{1}{3}}$$

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6. **(15 points)** Find the MacLaurin series for $\dfrac{x^2}{(1+2x)^3}$.

Solution. We have $(1-x)^{-1} = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$

Differentiating: $-(1-x)^{-2} \cdot (-1) = \displaystyle\sum_{n=0}^{\infty} nx^{n-1}$, i.e., $(1-x)^{-2} = \displaystyle\sum_{n=0}^{\infty} nx^{n-1}$

Differentiating again: $-2(1-x)^{-3} \cdot (-1) = \displaystyle\sum_{n=0}^{\infty} n(n-1)x^{n-2}$, so that $\dfrac{1}{(1-x)^3} = \displaystyle\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^{n-2}$

Substituting $-2x$ gives $\dfrac{1}{(1+2x)^3} = \displaystyle\sum_{n=0}^{\infty} \frac{n(n-1)}{2} \cdot (-2)^{n-2} x^{n-2}$

Multiplying by x^2 gives $\dfrac{x^2}{(1+2x)^3} = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{n(n-1)}{2} \cdot (-2)^{n-2} x^n}$

Note: this sum can also be written as $\displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot 2^{n-3} \cdot n(n-1)x^n$ or, since the $n = 0, 1$ terms are zero, as $\displaystyle\sum_{n=2}^{\infty} (-1)^n \cdot 2^{n-3} \cdot n(n-1)x^n$, or various other equivalent ways.

OPTIONAL BONUS A. (2 points.) Compute the **sum** of $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{4^n}$.

Solution. We have $(1-x)^{-1} = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$

Differentiating: $-(1-x)^{-2} \cdot (-1) = \sum_{n=0}^{\infty} nx^{n-1}$, i.e., $(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$

Multiplying by x gives $\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$

Differentiating again: the left side is $\dfrac{1(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \dfrac{1-x+2x}{(1-x)^3} = \dfrac{1+x}{(1-x)^3}$

and the right side is $\sum_{n=1}^{\infty} n^2 x^{n-1}$

So multiplying by x again: $\sum_{n=1}^{\infty} n^2 x^n = \dfrac{x(1+x)}{(1-x)^3}$

Plugging in $x = \dfrac{1}{4}$, we have: $\sum_{n=1}^{\infty} \dfrac{n^2}{4^n} = \dfrac{\frac{1}{4}\left(\frac{5}{4}\right)}{\left(\frac{3}{4}\right)^3} = \dfrac{1 \cdot 5 \cdot 4}{3^3} = \boxed{\dfrac{20}{27}}$

OPTIONAL BONUS B. (1 point.) The 2024 UN Climate Change Conference, also known as COP29, occurred in mid-November. In what nation was this conference held?

Answer. Azerbaijan.