Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #2

October 30, 2019

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted. Do not access any webpages during this exam.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, $\sinh(\ln 3)$, or $\arctan(\sqrt{3})$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		40
2		8
3		8
4		20
5		24
Total		100

1. [40 Points] Compute the following integral. Justify your work.

(a) $\displaystyle\int_0^{e^3} \frac{1}{x\,[9+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_t^{e^3} \frac{1}{x\left[9+(\ln x)^2\right]}\,dx$

$\boxed{\begin{array}{l} w=\ln x \\ dw=\frac{1}{x}dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow w=\ln t \\ x=e^3 \Rightarrow w=\ln e^3=3 \end{array}}$

$$= \lim_{t\to 0^+} \int_{\ln t}^{3} \frac{1}{9+w^2}\,dw$$

$$= \lim_{t\to 0^+} \frac{1}{3}\arctan\left(\frac{w}{3}\right)\Bigg|_{\ln t}^{3}$$

$$= \lim_{t\to 0^+} \frac{1}{3}\left[\arctan\left(\frac{\overset{1}{\cancel{3}}}{\cancel{3}}\right) - \arctan\left(\frac{\overset{-\infty}{\ln t}}{3}\right)\right]$$

$\frac{\pi}{4}$ $\qquad$ $-\frac{\pi}{2}$

$$= \frac{1}{3}\left[\frac{\pi}{4}+\frac{\pi}{2}\right] = \frac{1}{3}\left[\frac{3\pi}{4}\right] = \boxed{\frac{\pi}{4}} \quad \text{Converges}$$

1. (Continued) Compute the following integral. Justify your work.

(b) $\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 x \cdot \ln x \, dx$

$\boxed{\begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{array}}$

IBP

$$= \lim_{t \to 0^+} \frac{x^2}{2}\ln x \Big|_t^1 - \frac{1}{2}\int_t^1 \underbrace{x}_{\frac{1}{x}\cdot x^2} \, dx$$

$$= \lim_{t \to 0^+} \frac{x^2}{2}\ln x \Big|_t^1 - \frac{x^2}{4}\Big|_t^1$$

$$= \lim_{t \to 0^+} \frac{1}{2}\cdot \underbrace{\ln 1}_{0} - \underbrace{\frac{t^2}{2}\ln t}_{0^+\cdot(-\infty) \to 0\ (*)} - \left(\frac{1}{4} - \frac{\overset{\to 0}{t^2}}{4}\right)$$

See L'H below

$$= \boxed{\frac{-1}{4}} \quad \text{Converges.}$$

$(*)$ $\lim_{t \to 0^+} \underbrace{t^2 \cdot \ln t}_{0(-\infty)} = \lim_{t \to 0^+} \underbrace{\frac{\ln t}{\frac{1}{t^2}}}_{-\infty/\infty} \underset{L'H}{=} \lim_{t \to 0^+} \underbrace{\frac{\frac{1}{t}}{\frac{-2}{t^3}}}_{-t^3/2} = \lim_{t \to 0^+} \frac{-\overset{\to 0}{t^2}}{2} = 0$

$t^{-2} \to -2t^{-3}$

1. (Continued) Compute the following integral. Justify your work.

(c) $\int_7^9 \frac{10}{x^2-8x-9}\,dx = \int_7^9 \frac{10}{(x-9)(x+1)}\,dx = \lim_{t\to 9^-} \int_7^t \frac{10}{(x-9)(x+1)}\,dx$

Improper @ $x=9$

$= \lim_{t\to 9^-} \int_7^t \frac{1}{x-9} - \frac{1}{x+1}\,dx$

PFD

$-9)(x+1)$

$\left[\frac{10}{(x-9)(x+1)} = \frac{A}{x-9} + \frac{B}{x+1}\right](x-9)(x+1)$

$10 = A(x+1) + B(x-9)$

$= Ax + A + Bx - 9B$

$= (A+B)x + A - 9B$

Conditions

- $A+B=0 \quad B=-A$
- $A - 9B = 10$

$A - 9(-A) = 10$

$10A = 10$

$A = 1$

$\Rightarrow B = -1$

$= \lim_{t\to 9^-} \ln|x-9| - \ln|x+1| \Big|_7^t$

$= \lim_{t\to 9^-} \ln|t-9| - \ln|t+1| - [\ln|-2| - \ln 8]$

$\ln(10)$, 9^-, $|0^-|$, 0^+, $-\infty$, 2, $\ln 2$, finite

$= \boxed{-\infty}$ Diverges

1. (Continued) Compute the following integral. Justify your work.

(d) $\displaystyle\int_7^\infty \frac{10}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{10}{x^2-8x+19}\,dx$

$$= \lim_{t\to\infty}\int_7^t \frac{10}{(x-4)^2+3}\,dx$$

Complete Square

$x^2-8x+19 = (x-4)^2+3$

$\widehat{x^2-8x+16}+3$

$$\boxed{\begin{array}{l} w = x-4 \\ dw = dx \end{array}}$$

$$\boxed{\begin{array}{l} x=7 \Rightarrow w = 7-4 = 3 \\ x = t \Rightarrow w = t-4 \end{array}}$$

$$= \lim_{t\to\infty}\int_3^{t-4} \frac{10}{w^2+3}\,dw$$

$$= \lim_{t\to\infty} \frac{10}{\sqrt{3}} \arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_3^{t-4}$$

$$= \lim_{t\to\infty} \frac{10}{\sqrt{3}}\left[\overset{\frac{\pi}{2}}{\cancel{\arctan\left(\overset{\infty}{\cancel{\frac{t-4}{\sqrt{3}}}}\right)}} - \overset{\frac{\pi}{3}}{\cancel{\arctan\left(\overset{\sqrt{3}}{\cancel{\frac{3}{\sqrt{3}}}}\right)}}\right]$$

$$= \frac{10}{\sqrt{3}}\left[\underbrace{\frac{\pi}{2}-\frac{\pi}{3}}_{\frac{3\pi}{6}-\frac{2\pi}{6}}\right] = \frac{10}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{5\pi}{3\sqrt{3}}}$$

Converges.

2. [8 Points] Determine **and state** whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

$$\left\{ \left(\frac{n}{n+1}\right)^n \right\}_{n=1}^{\infty}$$

$\infty \cdot 0$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = e^{\lim_{x\to\infty} \ln\left[\left(\frac{x}{x+1}\right)^x\right]} = e^{\lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}} \ \tfrac{0}{0} \ \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{\left(\frac{x}{x+1}\right)} \cdot \left[\frac{(\cancel{x}+1)(1) - \cancel{x}(1)}{(x+1)^2}\right]}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{\cancel{x+1}}{\cancel{x}} \cdot \left(\frac{1}{(x+1)^{\cancel{2}}}\right)(-x^{\cancel{2}})}$$

$$= e^{\lim_{x\to\infty} \frac{-x}{x+1}} \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\cancel{-1}^{\,-1}}{\cancel{1}}} = e^{-1} = \boxed{\frac{1}{e}}$$

Sequence Converges

3. [8 Points] Find the **sum** of the following series (which does converge).

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{2n-1}}{4^{2n+1}}$$

$n=1 \quad n=2 \quad n=3$

$$= -\frac{3}{4^3} + \frac{3^3}{4^5} - \frac{3^5}{4^7} + \ldots$$

$$r = \frac{-3^2}{4^2}$$

$$a = \frac{-3}{4^3} = \frac{-3}{64}$$

$$r = \frac{-3^2}{4^2} = \frac{-9}{16}$$

Not Needed [Convergent Geometric Series $|r| = \left|\frac{-9}{16}\right| = \frac{9}{16} < 1$]

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{-3}{64}}{1-\left(-\frac{9}{16}\right)} = \frac{\frac{-3}{\cancel{64}_4}}{\frac{25}{16}} \ \frac{\cancel{16}}{25} = \boxed{-\frac{3}{100}}$$

$$\frac{16}{16} + \frac{9}{16}$$

4. [20 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2(n^3+1)}{n^3+1}$ $\xrightarrow{\text{A.S.}}$ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3+1} \approx \sum \frac{1}{n^3}$ Converges, p-Series $p=3>1$

Bound Terms

$$\frac{\sin^2(n^3+1)}{n^3+1} \leq \frac{\sin^2(n^3+1)}{n^3} \leq \frac{1}{n^3}$$ and

$\Rightarrow$ A.S. Converges by C.T.

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

Not Needed

(note: O.S. also A.C. by definition)

4. (Continued) Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(b) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{e} + \frac{1}{e^n}}_{a_n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e} + \cancel{\frac{1}{e^n}}^{\to 0} = \frac{1}{e} \neq 0$$

(c) $\sum_{n=1}^{\infty} \frac{e}{n^e} + \frac{1}{e^n}$ SPLIT

$$e\sum \frac{1}{n^e} + \sum \frac{1}{e^n}$$

Constant Multiple of Convergent p-Series $p = e > 1$ is Convergent

+

Convergent Geometric Series $|r| = \left|\frac{1}{e}\right| = \frac{1}{e} < 1$

or by GST

O.S. $\boxed{\text{Converges}}$ b/c "Sum of 2 Convergent Series Converges"

5. [24 Points] Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+9}{n^9+2}$ $\xrightarrow{\text{A.S.}}$ $\sum \frac{n^2+9}{n^9+2} \approx \sum \frac{n^2}{n^9} = \sum \frac{1}{n^7}$ Convergent p-Series $p=7>1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+9}{n^9+2}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{n^9+9n^7}{n^9+2}\frac{\left(\frac{1}{n^9}\right)}{\left(\frac{1}{n^9}\right)} = \lim_{n\to\infty} \frac{1+\cancel{\frac{9}{n^2}}^{\to 0}}{1+\cancel{\frac{2}{n^9}}^{\to 0}} = 1$$

Finite Non-zero

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{A.C.}}$ (by definition)

5. (Continued) Determine whether the given series is **absolutely convergent, conditionally convergent,** or **diverges.** Name any convergence test(s) you use, and justify all of your work.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(2n)!\,\ln n}{(n^n)\,n!}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(2(n+1))!\,\ln(n+1)}{(n+1)^{n+1}\cdot(n+1)!}}{\dfrac{\cancel{(-1)^n}\,(2n)!\,\ln n}{n^n\cdot n!}}\right|$$

$$= \lim_{n\to\infty} \frac{\overbrace{(2n+2)!}^{(2n+2)(2n+1)\cancel{(2n)!}}}{\cancel{(2n)!}}\left[\frac{\ln(n+1)}{\ln n}\right]\cdot\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}}\cdot\left[\frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}}\right]$$

(*) See Below

1

$\frac{1}{e}$

Don't Drop

$2(n+1)$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(\frac{1}{n})}{\cancel{(n+1)}(n+1)(\frac{1}{n})}\cdot\frac{1}{e} = \lim_{n\to\infty} 2\cdot\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)\cdot\frac{1}{e} = \frac{4}{e} > 1$$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by R.T.

(*) $$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

5. (Continued) Determine whether the given series is **absolutely convergent, conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+6}$ $\xrightarrow{\text{A.S.}}$ $\sum \frac{1}{n+6} \approx \sum \frac{1}{n}$ Diverges $\underset{\text{Harmonic}}{}$ p-Series $p=1$

Harmonic

Limit

$$\lim_{n\to\infty} \frac{\frac{1}{n+6}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\left(\frac{1}{n}\right)}{n+6\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{1+\cancel{\frac{6}{n}}_{\to 0}} = 1 \quad \left\{\begin{matrix}\text{Finite} \\ \text{Non-Zero}\end{matrix}\right\}$$

$\Rightarrow$ A.S. also Diverges by LCT

Try A.S.T O.S.

① $b_n = \frac{1}{n+6} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1^{\to 0}}{\underbrace{n+6}_{\to\infty}} = 0$

③ $b_{n+1} = \frac{1}{n+7} \leq \frac{1}{n+6} = b_n$

Terms Decreasing

O.S. Converges by A.S.T.

$\Rightarrow$ O.S. $\boxed{\text{C.C.}}$ (by definition)

****

OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

****

OPTIONAL BONUS #1 Compute the sum for this series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n+2}-e^{2n}}{(e^{2n}+1)(e^{2n+2}+1)}$

"slip-in/slip-out"

$$= \sum_{n=1}^{\infty} \frac{(e^{2n+2}+1)-(e^{2n}+1)}{(e^{2n}+1)(e^{2n+2}+1)} = \sum_{n=1}^{\infty} \frac{\cancel{(e^{2n+2}+1)}}{(e^{2n}+1)\cancel{(e^{2n+2}+1)}} - \frac{\cancel{(e^{2n}+1)}}{\cancel{(e^{2n}+1)}(e^{2n+2}+1)}$$

Telescoping

$$= \sum_{n=1}^{\infty} \frac{1}{e^{2n}+1} - \frac{1}{e^{2n+2}+1} = \underbrace{\left(\frac{1}{e^2+1} - \cancel{\frac{1}{e^4+1}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{e^4+1}} - \cancel{\frac{1}{e^6+1}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{e^6+1}} - \frac{1}{e^8+1}\right)}_{n=3} + \cdots + \left(\frac{1}{e^{2n}+1} - \frac{1}{e^{2n+2}+1}\right) + \cdots$$

$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{e^2+1} - \cancel{\frac{1}{e^{2n+2}+1}}^{\to 0} = \boxed{\frac{1}{e^2+1}}$$

Converges using Series Definition
$\sum a_n = \lim_{n\to\infty} S_n$

OPTIONAL BONUS #2 It can be shown that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

Compute the following sum. Justify. $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$

Given $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \cancel{\frac{1}{2}} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cancel{\frac{1}{6}} + \frac{1}{7} \ldots = \ln 2$

ADD

$$\Rightarrow \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\right) = \cancel{\frac{1}{2}} - \frac{1}{4} + \cancel{\frac{1}{6}} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \ldots = \frac{1}{2}\ln 2$$

Add Together

$$1 + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{7} - \frac{2}{8} + \cdots = \ln 2 + \frac{1}{2}\ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \boxed{\frac{3}{2}\ln 2}$$

Sum of 2 Convergent Series is Convergent