Math 121 Midterm Exam #2 November 2, 2016

1. [40 Points] For each of the following integrals, either compute it or show that it diverges.

(a) $\displaystyle\int_5^\infty \frac{1}{x^2-6x+13}\,dx = \lim_{t\to\infty}\int_5^t \frac{1}{x^2-6x+13}\,dx$

$\displaystyle= \lim_{t\to\infty}\int_5^t \frac{1}{(x-3)^2+4}\,dx \overset{\text{complete the}}{\text{square}}$

Substitute $\boxed{\begin{array}{l} u = x-3 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x=5 \Rightarrow u=2 \\ x=t \Rightarrow u=t-3 \end{array}}$

$\displaystyle= \lim_{t\to\infty}\int_2^{t-3}\frac{1}{u^2+4}\,du = \lim_{t\to\infty}\frac{1}{2}\arctan\left(\frac{u}{2}\right)\Bigg|_2^{t-3}$

$\displaystyle= \lim_{t\to\infty}\frac{1}{2}\left(\arctan\left(\frac{t-3}{2}\right) - \arctan 1\right) = \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{8}}$

(b) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\ln x\,dx = \lim_{t\to 0^+}\int_t^1 x^{-\frac{1}{2}}\ln x\,dx$

$\displaystyle= \lim_{t\to 0^+} 2\sqrt{x}\,\ln x\Big|_t^1 - 2\int_t^1 x^{-\frac{1}{2}}\,dx$

$\displaystyle= \lim_{t\to 0^+} 2\sqrt{x}\,\ln x\Big|_t^1 - 4\sqrt{x}\Big|_t^1 = \lim_{t\to 0^+} 2\ln 1 - 2\sqrt{t}\,\ln t - 4\left(1-\sqrt{t}\right)$

$\displaystyle= \lim_{t\to 0^+} 0 - 2\sqrt{t}\,\ln t - 4 \overset{(*)}{=} 0 - 4 = \boxed{-4}$ Converges

$(*)$ $\displaystyle\lim_{x\to 0^+}\sqrt{x}\,\ln x^{0\cdot-\infty} = \lim_{x\to 0^+}\frac{\ln x}{x^{-\frac{1}{2}}} \overset{\text{L'H}}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{\frac{3}{2}}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$

Integration By Parts:

$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-\frac{1}{2}}dx \\ du = \frac{1}{x}dx & v = 2\sqrt{x} \end{array}}$

(c) $\displaystyle\int_1^2 \frac{1}{x^2-x}\,dx = \int_1^2 \frac{1}{x(x-1)}\,dx$

$\displaystyle= \lim_{t\to 1^+}\int_t^2 \frac{1}{x(x-1)}\,dx \overset{\text{PFD}}{=} \lim_{t\to 1^+}\int_t^2 \frac{1}{x-1} - \frac{1}{x}\,dx$

$\displaystyle= \lim_{t\to 1^+} \ln|x-1| - \ln|x|\Big|_t^2 = \lim_{t\to 1^+} \ln 1 - \ln 2 - (\ln|t-1| - \ln|t|) = 0 - \ln 2 - (-\infty) + 0 = \boxed{\infty}$ Diverges

Partial Fractions Decomposition (PFD):

$\dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

Clearing the denominator yields:

$1 = A(x-1) + Bx$
$1 = (A+B)x - A$
so that $\quad A+B=0, \text{ and } -A=1$
Solve for $A=-1, \text{ and } B=11$

2. [10 Points] Determine **and state** whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

$$\left\{ \left(1+\ln\left(1+\frac{5}{n}\right)\right)^n \right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}\left(1+\ln\left(1+\frac{5}{n}\right)\right)^n = \lim_{x\to\infty}\left(1+\ln\left(1+\frac{5}{x}\right)\right)^x \quad 1^\infty$$

$$= e^{\lim\limits_{x\to\infty} \ln\left[\left(1+\ln\left(1+\frac{5}{x}\right)\right)^x\right]} = e^{\lim\limits_{x\to\infty} x\ln\left(1+\ln\left(1+\frac{5}{x}\right)\right)} \quad \infty\cdot 0$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ln\left(1+\ln\left(1+\frac{5}{x}\right)\right)}{\frac{1}{x}}} \overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{1}{1+\ln\left(1+\frac{5}{x}\right)}\right)\left(\frac{1}{1+\frac{5}{x}}\right)\left(-\frac{5}{x^2}\right)}{-\frac{1}{x^2}}} = \boxed{e^5} \text{ Converges}$$

3. [10 Points] Find the **sum** of the following series (which does converge).

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{4^{2n+1}}{3^{3n-1}} = -\frac{4^3}{3^2}+\frac{4^5}{3^5}-\frac{4^7}{3^8}+\dots$$

Here we have a geometric series with $a = -\dfrac{64}{9}$ and $r = -\dfrac{4^2}{3^3} = -\dfrac{16}{27}$

As a result, the sum is given by $\dfrac{a}{1-r} = \dfrac{-\frac{64}{9}}{1-\left(-\frac{16}{27}\right)} = \dfrac{-\frac{64}{9}}{\frac{43}{27}} = -\dfrac{64}{9}\cdot\dfrac{27}{43} = \boxed{-\dfrac{192}{43}}$

4. [15 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{\ln n}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

since $\lim\limits_{n\to\infty} \dfrac{n^2}{\ln n} = \lim\limits_{x\to\infty} \dfrac{x^2}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{=}{\text{L}'H}} \lim\limits_{x\to\infty} \dfrac{2x}{\frac{1}{x}} = \lim\limits_{x\to\infty} 2x^2 = \infty \neq 0$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2(5n)}{n^5+3}$

First examine the absolute series

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(5n)}{n^5+3}$

Next bound the terms

$\dfrac{\sin^2(5n)}{n^5+3} < \dfrac{1}{n^5+3} < \dfrac{1}{n^5}$ and

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series with $p = 5 > 1$.

Therefore, A.S. Convergent (A.C.) by CT. Finally the O.S. $\boxed{\text{Convergent}}$ by ACT.

5. [25 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n^5+6n^3}{n^9+4}$

First examine the absolute series

$\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+6n^3}{n^9+4}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+6n^3}{n^9+4} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which is a convergent p-series with $p = 4 > 1$.

Next,

Check: $\lim\limits_{n\to\infty} \dfrac{\dfrac{3n^5+6n^3}{n^9+4}}{\dfrac{1}{n^4}} = \lim\limits_{n\to\infty} \dfrac{3n^9+6n^7}{n^9+4} \dfrac{\left(\dfrac{1}{n^9}\right)}{\left(\dfrac{1}{n^9}\right)} = \lim\limits_{n\to\infty} \dfrac{3+\dfrac{6}{n}}{1+\dfrac{4}{n^9}} = 3$ which is finite and non-zero.

Therefore, these two series share the same behavior, and the absolute series is also Convergent, by Limit Comparison Test (LCT). Or more simply, A.S. CONV by LCT. Finally, we have $\boxed{\text{Absolute Convergence (A.C.)}}$

(Not needed here but **Note**: This implies that the Original Series is Convergent by ACT.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(n!)^2\,2^{4n}\,n^n}{(3n)!\,\ln n}$

Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}((n+1)!)^2\ 2^{4(n+1)}(n+1)^{n+1}}{(3(n+1))!\ln(n+1)}}{\dfrac{(-1)^n(n!)^2\ 2^{4n}n^n}{(3n)!\ln n}}\right|$$

$$= \lim_{n\to\infty}\frac{((n+1)!)^2}{(n!)^2}\cdot\frac{2^{4n+4}}{2^{4n}}\cdot\frac{(n+1)^{n+1}}{n^n}\cdot\frac{(3n)!}{(3n+3)!}\cdot\frac{\ln n}{\ln(n+1)}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3(16)}{(3n+3)(3n+2)(3n+1)}\cdot\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\frac{(n+1)^3(16)}{3(n+1)(3n+2)(3n+1)}\cdot\frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty}\frac{16e}{3}\cdot\left(\frac{n+1}{3n+2}\right)\left(\frac{n+1}{3n+1}\right)$$

$$= \lim_{n\to\infty}\frac{16e}{3}\cdot\left(\frac{1+\frac{1}{n}}{3+\frac{2}{n}}\right)\left(\frac{1+\frac{1}{n}}{3+\frac{1}{n}}\right) = \frac{16e}{27} > 1$$ The original series $\boxed{\text{Diverges by the Ratio Test}}$.

Or more simply, O.S. DIVERGES by R.T.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\ \frac{1}{\sqrt{n}+1}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1} \approx \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is a divergent p-series with $p = \dfrac{1}{2} < 1$. Next,

Check: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+1}\frac{\left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{\sqrt{n}}} = 1$ which is finite and non-zero.

Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is the divergent p-Series $\left(p = \dfrac{1}{2} < 1\right)$, then the absolute series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$ is also divergent by Limit Comparison Test. Or more simply, A.S. DIV by LCT. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{1}{\sqrt{n}+1} > 0$

- $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = 0$

- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$ because $b_{n+1} = \dfrac{1}{\sqrt{n+1}+1} < \dfrac{1}{\sqrt{n}+1} = b_n$

OR to show terms decreasing, could also show that for $f(x) = \dfrac{1}{\sqrt{x}+1}$, we have $f'(x) = -\left(\dfrac{1}{2\sqrt{x}(\sqrt{x}+1)^2}\right) < 0$.

Therefore, the original series converges by the Alternating Series Test. (Or simply O.S. CONV by AST) Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent (C.C.)}}$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the sum of the following series

$$\sum_{n=2}^{\infty} \frac{e^{2n+2}-e^{2n}}{(e^{2n}+1)(e^{2n+2}+1)} = \sum_{n=2}^{\infty} \frac{e^{2n+2}+1-(e^{2n}+1)}{(e^{2n}+1)(e^{2n+2}+1)}$$

$$= \sum_{n=2}^{\infty} \frac{e^{2n+2}+1}{(e^{2n}+1)(e^{2n+2}+1)} - \frac{e^{2n}+1}{(e^{2n}+1)(e^{2n+2}+1)} = \sum_{n=2}^{\infty} \frac{1}{e^{2n}+1} - \frac{1}{e^{2n+2}+1}$$

$$= \left(\frac{1}{e^4+1} - \frac{1}{e^6+1}\right) + \left(\frac{1}{e^6+1} - \frac{1}{e^8+1}\right) + \dots$$

This series looks telescoping. Examine the partial sum:

$$S_n = \left(\frac{1}{e^4+1} - \frac{1}{e^6+1}\right) + \left(\frac{1}{e^6+1} - \frac{1}{e^8+1}\right) + \dots + \left(\frac{1}{e^{2n}+1} - \frac{1}{e^{2n+2}+1}\right)$$

$$\text{O.S. } \sum_{n=1}^{\infty} \frac{e^{2n+2}-e^{2n}}{(e^{2n}+1)(e^{2n+2}+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{e^4+1} - \frac{1}{e^{2n+2}+1} = \boxed{\frac{1}{e^4+1}}.$$

OPTIONAL BONUS #2 Compute the following integral $\displaystyle\int \frac{x^5+7x^3+x^2+13x+2}{x^4+6x^2+9}\,dx = \int x + \frac{x^3+x^2+4x+2}{(x^2+3)^2}\,dx$

$$= \int x + \frac{x+1}{x^2+3} + \frac{x-1}{(x^2+3)^2}\,dx = \int x + \frac{x}{x^2+3} + \frac{1}{x^2+3} + \frac{x}{(x^2+3)^2} - \frac{1}{(x^2+3)^2}\,dx$$

(**see below for last piece)

$$\boxed{= \frac{x^2}{2} + \frac{1}{2}\ln|x^2+3| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2(x^2+3)} - \frac{\sqrt{3}}{18}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{x}{6(x^2+3)} + C}$$

Long division yields:

$$x^4+6x^2+9\overline{\smash{)}x^5+7x^3+x^2+13x+2}^{\;x}$$

$$\underline{-(x^5+6x^3+9x)}$$

$$x^3+x^2+4x+2$$

Partial Fractions Decomposition:

$$\frac{x^3+x^2+4x+2}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}$$

Clearing the denominator yields:

$$\begin{aligned}
&x^3+x^2+4x+2 = (Ax+B)(x^2+3)+(Cx+D)\\
&x^3+x^2+4x+2 = (Ax+B)(x^2+3)+Cx+D\\
&x^3+x^2+4x+2 = Ax^3+Bx^2+3Ax+3B+Cx+D\\
&x^3+x^2+4x+2 = Ax^3+Bx^2+(3A+C)x+(3B+D)
\end{aligned}$$

so that $A=1,\ B=1,\ 3A+C=4$ and $3B+D=2$

Solve for $A=1,\ B=1,\ C=1$ and $D=-1$

$$(**)\int \frac{1}{(x^2+3)^2}\ dx = \int \frac{1}{((\sqrt{3}\tan\theta)^2+3)^2}\sqrt{3}\sec^2\theta\ d\theta = \int \frac{1}{(3\tan^2\theta+3)^2}\sqrt{3}\sec^2\theta\ d\theta$$

$$= \frac{\sqrt{3}}{9}\int \frac{\sec^2\theta}{\sec^4\theta}\ d\theta = \frac{\sqrt{3}}{9}\int \frac{1}{\sec^2\theta}\ d\theta = \frac{\sqrt{3}}{9}\int \cos^2\theta\ d\theta = \frac{\sqrt{3}}{9}\int \frac{1+\cos(2\theta)}{2}\ d\theta$$

$$= \frac{\sqrt{3}}{18}\int 1+\cos(2\theta)\ d\theta = \frac{\sqrt{3}}{18}\left(\theta+\frac{\sin(2\theta)}{2}\right)+C = \frac{\sqrt{3}}{18}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+C$$

$$= \frac{\sqrt{3}}{18}\left(\theta+\sin\theta\cos\theta\right)+C$$

$$= \frac{\sqrt{3}}{18}\left(\arctan\left(\frac{x}{\sqrt{3}}\right)+\frac{x}{\sqrt{x^2+3}}\cdot\frac{\sqrt{3}}{\sqrt{x^2+3}}\right)+C = \frac{\sqrt{3}}{18}\left(\arctan\left(\frac{x}{\sqrt{3}}\right)+\frac{\sqrt{3}x}{x^2+3}\right)+C$$

$$= \frac{\sqrt{3}}{18}\arctan\left(\frac{x}{\sqrt{3}}\right)+\frac{\sqrt{3}}{18}\left(\frac{\sqrt{3}x}{x^2+3}\right)+C = \frac{\sqrt{3}}{18}\arctan\left(\frac{x}{\sqrt{3}}\right)+\frac{3x}{18(x^2+3)}+C$$

$$= \frac{\sqrt{3}}{18}\arctan\left(\frac{x}{\sqrt{3}}\right)+\frac{x}{6(x^2+3)}+C$$

Trig. Substitute $\boxed{\begin{aligned} x &= \sqrt{3}\tan\theta \\ dx &= \sqrt{3}\sec^2\theta d\theta \end{aligned}}$