Math 121-03, Fall 2024 Professor Rob Benedetto

Solutions to Midterm Exam 2

1. **(20 points, 2 parts)** Compute the following **improper** integrals. Justify your work.

1a. $\displaystyle\int_{-\infty}^{5} \frac{1}{x^2-6x+13}\,dx$ 1b. $\displaystyle\int_{-1}^{2} \frac{6}{x^2+2x-8}\,dx$

$\left(\textit{Hint for } b\text{: You may use this } \textbf{free} \text{ fact: } \dfrac{6}{x^2+2x-8} = \dfrac{1}{x-2} - \dfrac{1}{x+4} \right)$

Solution. 1a. $\displaystyle\int_{-\infty}^{5} \frac{1}{x^2-6x+13}\,dx = \lim_{t\to-\infty}\int_{t}^{5} \frac{1}{x^2-6x+13}\,dx = \lim_{t\to-\infty}\int_{t}^{5} \frac{1}{(x-3)^2+4}\,dx$

$[u = x-3, du = dx]$

$$= \lim_{t\to-\infty}\int_{t-3}^{2} \frac{1}{u^2+4}\,du = \lim_{t\to-\infty} \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Bigg|_{t-3}^{2} = \lim_{t\to-\infty}\frac{1}{2}\left[\arctan 1 - \arctan\left(\frac{t-3}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \big(\arctan(-\infty)\big)\right) = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \boxed{\frac{3\pi}{8}}$$

1b. $\displaystyle\int_{-1}^{2} \frac{6}{x^2+2x-8}\,dx = \lim_{t\to2^-}\int_{-1}^{t} \frac{1}{x-2} - \frac{1}{x+4}\,dx = \lim_{t\to2^-}\left(\ln|x-2| - \ln|x+4|\Big|_{-1}^{t}\right)$

$$= \lim_{t\to2^-}\Big(\big(\ln|t-2| - \ln|t+4|\big) - \big(\ln|-3| - \ln|3|\big)\Big) = \lim_{t\to2^-}\ln\left|\frac{t-2}{t+4}\right| = \ln 0 = -\infty,$$

so the integral $\boxed{\textbf{diverges}}$

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2. **(10 points)** Use the $\boxed{\textbf{integral test}}$ to determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{7n+2}$ converges or diverges.

(You may skip checking the three preconditions for using the integral test.)

Solution. Consider the associated integral:

$\displaystyle\int_{1}^{\infty}\frac{1}{7x+2}\,dx = \lim_{t\to\infty}\int_{1}^{t}\frac{1}{7x+2}\,dx \quad [u = 7x+2, du = 7\,dx, dx = \frac{1}{7}\,du]$

$$= \lim_{t\to\infty}\frac{1}{7}\int_{9}^{7t+2}\frac{1}{u}\,du = \lim_{t\to\infty}\frac{1}{7}\ln|u|\Big|_{9}^{7t+2} = \lim_{t\to\infty}\frac{1}{7}\big(\ln|7t+2| - \ln 9\big)$$

$= \dfrac{1}{7}(\infty - \ln 9) = \infty$, so the integral divereges.

Therefore, by the integral test, the O.S. $\boxed{\text{diverges}}$

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3. **(30 points, 3 parts)** Determine whether each of the following series **converges or diverges**. Name any convergence test(s) you use (abbreviations are OK), and justify all of your work.

3a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^{5e}} + \frac{1}{5e^n}\right)$ 3b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{\ln(n+2)}$ 3c. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sin^2(5n)}{n^5+2}$

(*Suggestion for c*: consider the absolute series.)

Solution. 3a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^{5e}} + \frac{1}{5e^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^{5e}} + \sum_{n=1}^{\infty}\frac{1}{5e^n}$ by arithmetic of series.

The first summand, $\sum \frac{1}{n^{5e}}$, converges by the p-Test with $p = 5e > 1$.

The second summand, $\sum \frac{1}{5e^n}$ is geometric, with $r = \frac{1}{e}$, and so converges by GST since $|r| = \frac{1}{e} < 1$.

So the O.S. $\boxed{\textbf{converges}}$ as the sum of two convergent series.

3b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\ln(n+2)}$ has:

$$\lim_{n\to\infty} \frac{n^2}{\ln(n+2)} = \lim_{x\to\infty} \frac{x^2}{\ln(x+2)} \overset{(\frac{\infty}{\infty})^{\text{L'H}}}{=} \lim_{x\to\infty} \frac{2x}{\frac{1}{x+1}} = \lim_{x\to\infty} 2x(1+x) = \infty \neq 0.$$

Therefore, the O.S. $\boxed{\textbf{diverges} \text{ by nTDT}}$

3c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2(5n)}{n^5+2}$ has absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(5n)}{n^5+2}$

We have $0 \leq \frac{\sin^2(5n)}{n^5+2} \leq \frac{1}{n^5+2} \leq \frac{1}{n^5}$.

We also have $\sum \frac{1}{n^5}$ converges by the p-Test, with $p = 5 > 1$.

So the A.S. converges by CT.

So the O.S. $\boxed{\textbf{converges} \text{ by ACT}}$

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4. **(40 points, 3 parts, 3 pages)** Determine whether each of the following series is **Absolutely Convergent**, is **Conditionally Convergent**, or **Diverges**. Name any convergence test(s) you use (abbreviations are OK), and justify all of your work.

4a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n+2}$ 4b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot n!}{n^2 \cdot n^n}$ 4c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5n}{n^5+2}$

Solution. 4a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n+2}$ has absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2}$.

We have $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = \frac{1}{1+0} = 1 > 0$ and finite.

Now $\sum \frac{1}{n}$ diverges by p-Test with $p = 1$. So the A.S. diverges by LCT.

MEANWHILE, the O.S. is alternating, with:

- $\frac{1}{n+2} > 0$,
- $\displaystyle\lim_{n\to\infty} \frac{1}{n+2} = \frac{1}{\infty} = 0$, and
- $\frac{1}{(n+1)+2} > \frac{1}{n+2}$ is decreasing.

So the O.S. converges by AST.

So the O.S. is $\boxed{\textbf{Conditionally Convergent}}$ by definition.

4b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot n!}{n^2 \cdot n^n}$

Use RT: $L = \displaystyle\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}2^{n+1}(n+1)!}{(n+1)^2(n+1)^{n+1}}}{\frac{(-1)^n 2^n (n)!}{n^2 \cdot n^n}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \left(\frac{n}{n+1}\right)^2 \cdot \frac{n^n}{(n+1)^{n+1}}$

$$= \lim_{n\to\infty} 2\frac{\cancel{(n+1)n!}}{\cancel{n!}} \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^2 \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{\cancel{n+1}} = 2 \cdot (1)^2 \cdot \frac{1}{e} = \frac{2}{e} < 1.$$

So O.S. is $\boxed{\textbf{Absolutely Convergent}}$ by RT.

4c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5n}{n^5+2}$ has absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+5n}{n^5+2}$.

We have $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2+5n}{n^5+2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^5+5n^4}{n^5+2} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{1+\frac{2}{n^5}} = \frac{1+0}{1+0} = 1 > 0$ and finite.

Now $\displaystyle\sum \frac{1}{n^3}$ converges by p-Test with $p = 3 > 1$. So the A.S. converges by LCT.

So the O.S. is $\boxed{\textbf{Absolutely Convergent}}$ by definition.

OPTIONAL BONUS A. (2 points.) Define a sequence $\{b_n\}_{n=1}^{\infty}$ by $b_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is odd} \\ \frac{1}{2n} & \text{if } n \text{ is even.} \end{cases}$

Determine whether $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ **converges or diverges**. Name any convergence test(s) you use (abbreviations are OK), and justify all of your work.

Solution. Suppose $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ converged. Then grouping pairs of terms, we have:

$$\sum_{n=1}^{\infty} (-1)^n b_n = -\frac{1}{2^1} + \frac{1}{2(2)} - \frac{1}{2^3} + \frac{1}{2(4)} - \frac{1}{2^5} + \frac{1}{2(6)} - \frac{1}{2^7} + \frac{1}{2(8)} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{2^{2n-1}} + \frac{1}{4n} \right)$$

Note that the series $\displaystyle\sum_{n=1} \frac{1}{2^{2n-1}}$ is a geometric series with ratio $r = \frac{1}{4}$, which has $|r| < 1$. Thus, this geometric series converges by the geometric series test. Therefore, by the arithmetic of series,

$$\sum_{n=1}^{\infty} \frac{1}{4n} = \sum_{n=1}^{\infty} \left[\frac{1}{2^{2n-1}} + \left(-\frac{1}{2^{2n-1}} + \frac{1}{4n} \right) \right] = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} + \sum_{n=1}^{\infty} (-1)^n b_n$$

is a sum of two convergent series and hence converges. However, $\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-Test, with $p = 1$.

This is a contradiction, so our original supposition must be false.

That is, $\boxed{\text{the series } \sum_{n=1}^{\infty} (-1)^n b_n \text{ diverges}}$

OPTIONAL BONUS B. (1 point.) You may have heard that there is a US presidential election coming up. Americans do not vote directly for the president, but instead for members of the Electoral College; these members, called Electors, then in turn cast votes for president. How many Electors are there in the US Electoral College?

Answer. $\boxed{538}$ (which is where "FiveThirtyEight.com" gets its name.)

Each state gets one Elector for each Senator (two per state, totalling 100), and one for each member of the House of Representatives (totalling 435 across all 50 states, with larger-population states having more Representatives). Finally, Washington, DC gets three Electors, the same as the states with the minimum number of Electors.