Solutions to Midterm Exam 1

1. **(8 points)** Evaluate the following limit, showing all your steps: $\lim_{x\to 0^+} \sqrt{x}\ln x$

Solution. $\overset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\left(\frac{-\infty}{\infty}\right)^{\text{L'H}}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = -2\sqrt{0} = \boxed{0}$

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2. **(14 points)** Evaluate the following limit, showing all your steps: $\lim_{x\to 0} \frac{\cos(3x) - \arcsin(2x) + 2x - 1}{e^{4x} - 1 - 4x}$

Solution. Plugging in gives $\frac{1-0+0-1}{1-1-0}$, so:

$$\overset{\left(\frac{0}{0}\right)^{\text{L'H}}}{=} \lim_{x\to 0} \frac{-3\sin(3x) - \frac{1}{\sqrt{1-4x^2}}(2) + 2}{4e^{4x} - 4} = \lim_{x\to 0} \frac{-3\sin(3x) - 2(1-4x^2)^{-1/2} + 2}{4e^{4x} - 4}$$

$$\overset{\left(\frac{0}{0}\right)^{\text{L'H}}}{=} \lim_{x\to 0} \frac{-9\cos(3x) + (1-4x^2)^{-3/2}(-8x)}{16e^{4x}} = \frac{-9(1) + 1(0)}{16(1)} = \boxed{-\frac{9}{16}}$$

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3. **(16 points)** Evaluate the following limit, showing all your steps: $\lim_{x\to\infty} \left(1 - \arctan\left(\frac{4}{x}\right)\right)^x$

Solution. $\overset{1^\infty}{=} \exp\left(\lim_{x\to\infty} x \ln\left(1 - \arctan\left(\frac{4}{x}\right)\right)\right) \overset{\infty\cdot 0}{=} \exp\left(\lim_{x\to\infty} \frac{\ln\left(1 - \arctan\left(\frac{4}{x}\right)\right)}{x^{-1}}\right)$

$$\overset{\left(\frac{0}{0}\right)^{\text{L'H}}}{=} \exp\left(\lim_{x\to\infty} \frac{\frac{1}{1-\arctan\left(\frac{4}{x}\right)} \cdot \frac{(-1)}{1+\left(\frac{4}{x}\right)^2} \cdot \left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}}\right)$$

$$= \exp\left(\lim_{x\to\infty} \frac{1}{1-\arctan\left(\frac{4}{x}\right)} \cdot \frac{(-1)}{1+\left(\frac{4}{x}\right)^2} \cdot (4)\right) = \exp\big(1\cdot(-1)\cdot 4\big) = \boxed{e^{-4}}$$

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4. **(18 points)** Compute the indefinite integral $\int x \arcsin x \, dx$

Solution.

$u = \arcsin x$	$dv = x\,dx$
$du = \frac{dx}{\sqrt{1-x^2}}$	$v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

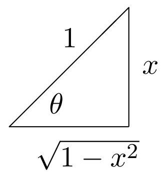
$x = \sin\theta$
$dx = \cos\theta\,d\theta$
$1 - x^2 = \cos^2\theta$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta\cos\theta\,d\theta}{\sqrt{\cos^2\theta}} = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1 - \cos 2\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

Where we used this trig triangle:



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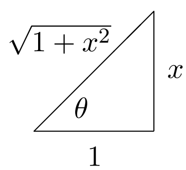
5. **(18 points)** Compute the indefinite integral $\displaystyle\int \frac{1}{(1+x^2)^{7/2}}\,dx$

Solution. $\boxed{\begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \\ 1+x^2 = \sec^2\theta \end{array}}$ $\displaystyle = \int \frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \int \cos^5\theta\,d\theta = \int (1-\sin^2\theta)^2\cos\theta\,d\theta$

$\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}}$ $\displaystyle = \int (1-u^2)^2\,du = \int 1 - 2u^2 + u^4\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$$



$$= \boxed{\frac{x}{\sqrt{1+x^2}} - \frac{2x^3}{3(1+x^2)^{3/2}} + \frac{x^5}{5(1+x^2)^{5/2}} + C}$$

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6. **(12 points)** Compute the definite integral $\displaystyle\int_0^3 xe^{-2x}\,dx$

Solution. $\boxed{\begin{array}{ll} u = x & dv = e^{-2x}\,dx \\ du = dx & v = -\frac{1}{2}e^{-2x} \end{array}}$ $\displaystyle = -\frac{x}{2}e^{-2x}\Big|_0^3 - \int_0^3 \left(-\frac{1}{2}\right)e^{-2x}\,dx$

$$= -\frac{3}{2}e^{-6} + 0 + \frac{1}{2}\int_0^3 e^{-2x}\,dx = -\frac{3}{2}e^{-6} + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\Big|_0^3\right] = -\frac{3}{2}e^{-6} - \frac{1}{4}\left[e^{-6} - 1\right] = \boxed{\frac{1}{4} - \frac{7}{4}e^{-6}}$$

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7. **(14 points)** Compute the definite integral $\displaystyle\int_e^{e^3} \frac{1}{x\left[3+(\ln x)^2\right]}\,dx$

Solution. $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{ll} x = e & \Rightarrow u = \ln e = 1 \\ x = e^3 & \Rightarrow u = \ln e^3 = 3 \end{array}}$ $\displaystyle = \int_1^3 \frac{du}{3+u^2} = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^3$

$$= \frac{1}{\sqrt{3}}\left[\arctan\left(\sqrt{3}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

OPTIONAL BONUS A. (2 points.) Compute the indefinite integral $\displaystyle\int \frac{e^x\,dx}{1-\sin\left(e^x\right)}$

Solution. $\boxed{\begin{array}{ll} u & = e^x \\ du & = e^x\,dx \end{array}}$ $\displaystyle = \int \frac{du}{1-\sin u} = \int \frac{(1+\sin u)\,du}{(1+\sin u)(1-\sin u)} = \int \frac{(1+\sin u)\,du}{1-\sin^2 u}$

$$= \int \frac{(1 + \sin u)\,du}{\cos^2 u} = \int \sec^2 u + \sec u \tan u \, du = \tan u + \sec u + C = \boxed{\tan(e^x) + \sec(e^x) + C}$$

OPTIONAL BONUS B. (1 point.) Less than two months ago, on August 5, 2024, the Prime Minister of Bangladesh resigned amid mass protests in that country. This longest-serving Prime Minister of Bangladesh had also faced accusations of authoritarianism. Name this (now former) Prime Minister.

Answer. Sheikh Hasina. She was prime minister from 1996 to 2001, and then again from 2009 to 2024.