Extra Examples of Integration By Parts Integrals

Integration by Parts is an integration technique that computes antiderivatives for some special products of a certain form. Here is the formula:

$$(*) \quad \int f(x) \cdot g'(x)\ dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)\ dx$$

We often use a shorthand notation

$$u = f(x) \Rightarrow du = f'(x)dx$$

$$v = g(x) \Rightarrow dv = g'(x)dx$$

Rewriting the formula (*) above, we have a formula to use as a memory aid:

$$\boxed{\int u\ dv = uv - \int v\ du}$$

To use this formula, follow these steps.

1. Choose a portion of the integral for the u piece so that it differentiates *nicely.*

2. Choose the rest of the integral for the dv piece so that it antidifferentiates *nicely.*

3. Look for the new integral on the right $\int v\ du$ to appear simpler to compute than the original integral.

To choose u, we use the Memory Aid **LIPET**. Once you label the product pieces, then match the formula above. Here, the better choices for u range in order from Left to Right.

$\overset{\longrightarrow}{L}$	I	$\overset{\longrightarrow}{P}$	E	$\overset{\longrightarrow}{T}$
o	n	o	x	r
g	v	l	p	i
	e	y	o	g
	r	n	n	
	s	o	e	
	e	m	n	
	T	i	t	
	r	a	i	
	i	l	a	
	g	s	l	
			s	

Here are some various examples. First study the integrals of some inverse functions:

1.

$$\int \ln x \, dx = \int (\ln x) \cdot 1 \, dx \overset{\text{I.B.P}}{=} x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = \boxed{x \ln x - x + C}$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = 1dx \\ du = \frac{1}{x}dx & v = x \end{array}}$$

2.

$$\begin{aligned} \int \arctan x \, dx &= \int (\arctan x) \cdot 1 \, dx \overset{\text{I.B.P}}{=} x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \int \frac{1}{u} \, du \\ &= x \arctan x - \frac{1}{2} \ln |u| + C = \boxed{x \arctan x - \frac{1}{2} \ln |1 + x^2| + C} \end{aligned}$$

$$\boxed{\begin{array}{ll} u = \arctan x & dv = 1dx \\ du = \frac{1}{1+x^2}dx & v = x \end{array}} \quad \boxed{\begin{array}{l} u = 1 + x^2 \\ du = 2xdx \\ \frac{1}{2}du = xdx \end{array}}$$

3.

$$\begin{aligned} \int \arcsin x \, dx &= \int (\arcsin x) \cdot 1 \, dx \overset{\text{I.B.P}}{=} x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \arcsin x - \left(-\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du \right) = x \arcsin x + \frac{1}{2} \int u^{-\frac{1}{2}} \, du \\ &= x \arcsin x + \frac{1}{2} \cdot 2\sqrt{u} + C = \boxed{x \arcsin x + \sqrt{1 - x^2} + C} \end{aligned}$$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = 1dx \\ du = \frac{1}{\sqrt{1-x^2}}dx & v = x \end{array}} \quad \boxed{\begin{array}{l} u = 1 - x^2 \\ du = -2xdx \\ -\frac{1}{2}du = xdx \end{array}}$$

4. Now Double Integration by Parts Example

$$\int \frac{x^2}{e^{5x}}\,dx = \int \overset{P}{x^2}\overset{E}{e^{-5x}}\,dx \overset{\text{I.B.P}}{=} -\frac{1}{5}x^2e^{-5x} - 2\left(-\frac{1}{5}\right)\int \overset{P}{x}\overset{E}{e^{-5x}}\,dx$$

$$= -\frac{1}{5}x^2e^{-5x} + \frac{2}{5}\int xe^{-5x}dx$$

$$\overset{\text{I.B.P}}{=} -\frac{1}{5}x^2e^{-5x} + \frac{2}{5}\left[-\frac{1}{5}xe^{-5x} - \left(-\frac{1}{5}\right)\int e^{-5x}\,dx\right]$$

$$= -\frac{1}{5}x^2e^{-5x} + \frac{2}{5}\left[-\frac{1}{5}xe^{-5x} + \left(\frac{1}{5}\right)\int e^{-5x}\,dx\right]$$

$$= -\frac{1}{5}x^2e^{-5x} + \frac{2}{5}\left[-\frac{1}{5}xe^{-5x} + \left(\frac{1}{5}\right)\left(-\frac{1}{5}\right)e^{-5x}\right] + C$$

$$= \boxed{-\frac{1}{5}x^2e^{-5x} - \frac{2}{25}xe^{-5x} - \frac{2}{125}e^{-5x} + C}$$

$u = x^2$	$dv = e^{-5x}dx$
$du = 2xdx$	$v = -\frac{1}{5}e^{-5x}$

$u = x$	$dv = e^{-5x}dx$
$du = dx$	$v = -\frac{1}{5}e^{-5x}$

5.

$$\int \overset{P}{x^2}\overset{T}{\cos}x\,\,dx \overset{\text{I.B.P}}{=} x^2\sin x - 2\int \overset{P}{x}\overset{T}{\sin}\,x\,dx$$

$$\overset{\text{I.B.P}}{=} x^2\sin x - 2\left(-x\cos x \overset{--}{+} \int \cos x\,dx\right)$$

$$= x^2\sin x + 2x\cos x - 2\int \cos x\,dx$$

$$= \boxed{x^2\sin x + 2x\cos x - 2\sin x + C}$$

$u = x^2$	$dv = \cos x dx$
$du = 2xdx$	$v = \sin x$

$u = x$	$dv = \sin x dx$
$du = dx$	$v = -\cos x$

6.

$$\int \overset{P}{x} \overset{I}{\arctan x}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C$$

$$= \boxed{\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C}$$

$u = \arctan x$	$dv = x dx$
$du = \frac{1}{1+x^2}dx$	$v = \frac{x^2}{2}$

7.

$$\int \frac{\ln x}{x^2}\ dx = \int (\overset{L}{\ln x})\cdot \overset{P}{x^{-2}}\ dx = -\frac{\ln x}{x} \overset{--}{+} \int x^{-2}\ dx = \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}$$

$u = \ln x$	$dv = x^{-2}dx$
$du = \frac{1}{x}dx$	$v = -\frac{1}{x}$

8.

$$\int \overset{P}{x^2}\overset{E}{e^x}\ dx \overset{\text{I.B.P}}{=} x^2e^x - 2\int \overset{P}{x}\overset{E}{e^x}\ dx = x^2e^x - 2\left(xe^x - \int e^x\ dx\right) = \boxed{x^2e^x - 2xe^x + 2e^x + C}$$

$u = x^2$	$dv = e^x dx$
$du = 2x dx$	$v = e^x$

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

Here is the formula for Integration by Part for Definite Integrals.

$$\boxed{\int_a^b u\ dv = uv\Big|_a^b - \int_a^b v\ du}$$

There is no need to change the Limits of Integration unless a u-substitution is used along the way.

9.

$$\int_0^1 \ln(x^2+1)\ dx \quad \int_0^1 (\ln(x^2+1)) \cdot 1\ dx$$

$$= x\ln(x^2+1)\Big|_0^1 - 2\int_0^1 \frac{x^2}{x^2+1}\ dx$$

$$= x\ln(x^2+1)\Big|_0^1 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\ dx$$

$$= x\ln(x^2+1)\Big|_0^1 - 2\left(\int_0^1 \frac{x^2+1}{x^2+1}\ dx - \int_0^1 \frac{1}{x^2+1}\ dx\right)$$

$$= x\ln(x^2+1)\Big|_0^1 - 2\left(\int_0^1 1\ dx - \int_0^1 \frac{1}{x^2+1}\ dx\right)$$

$$= x\ln(x^2+1)\Big|_0^1 - 2x\Big|_0^1 + 2\arctan x\Big|_0^1$$

$$= \ln 2 - \ln 1 - 2 + 0 + 2\arctan(1) - 2\arctan(0)$$

$$= \ln 2 - 0 - 2 + 2\left(\frac{\pi}{4}\right) - 0 = \boxed{\ln 2 - 2 + \frac{\pi}{2}}$$

I.B.P.

$$\boxed{\begin{array}{ll} u = \ln(x^2+1) & dv = dx \\ du = \dfrac{2x}{x^2+1}dx & v = x \end{array}}$$

10.

$$\int_1^e (\ln x)^2 \ dx \quad \int_1^e (\ln x)^2 \cdot 1 \ dx$$

$$= x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \cdot \left(\frac{1}{x}\right) \cdot x \ dx$$

$$= x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \ dx$$

$$= x(\ln x)^2\Big|_1^e - 2\left(x \ln x\Big|_1^e - \int_1^e \left(\frac{1}{x}\right) \cdot x \ dx\right)$$

$$= x(\ln x)^2\Big|_1^e - 2\left(x \ln x\Big|_1^e - \int_1^e 1 \ dx\right)$$

$$= x(\ln x)^2\Big|_1^e - 2\left(x \ln x\Big|_1^e - x\Big|_1^e\right)$$

$$= x(\ln x)^2\Big|_1^e - 2x \ln x\Big|_1^e + 2x\Big|_1^e$$

$$= (e(\ln e)^2 - 1(\ln 1)^2) - 2\,(e \ln e - 1 \ln 1) + 2\,(e-1)$$

$$= (e-0) - 2(e-0) + 2(e-1) = e - 2e + 2e - 2 = \boxed{e-2}$$

First I.B.P. $\boxed{u = (\ln x)^2 \quad dv = dx \quad du = 2\ln x \frac{1}{x} dx \quad v = x}$ Second I.B.P. $\boxed{u = \ln x \quad dv = dx \quad du = \frac{1}{x} dx \quad v = x}$

11.

$$\int_1^e \ln x \ dx \ = \int_1^e (\ln x) \cdot 1 \ dx \overset{\text{I.B.P}}{=} x \ln x\Big|_1^e - \int_1^e x \cdot \frac{1}{x} \ dx = x \ln x\Big|_1^e - \int_1^e 1 \ dx$$

$$= x \ln x\Big|_1^e - x\Big|_1^e = e \ln e - 1 \ln 1 - (e-1) = e - 0 - e + 1 = \boxed{1}$$

$$\boxed{u = \ln x \quad dv = dx \quad du = \frac{1}{x} dx \quad v = x}$$