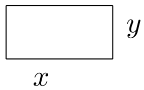
Math 111-01, Fall 2022 Professor Rob Benedetto

Solutions to Practice Problems 3

1. Show that of all rectangles with area 64, the one with the smallest perimeter is a square.

Solution. Diagram: 

The area is $xy = 64$, so $y = 64/x$.

The perimeter is then $P = 2x + 2y = 2x + 2 \cdot \frac{64}{x}$, and the common sense bounds $x > 0$ and $y > 0$ give just $x > 0$.

So we must minimize $P(x) = 2x + 2\frac{64}{x}$ on the domain $(0, \infty)$.

We compute $P'(x) = 2 - 128x^{-2}$, which is defined everywhere on $(0, \infty)$. Solving $P' = 0$ gives $x^2 = 64$, so $x = \pm 2$. We discard -8, which is not in the domain. So $x = 8$ is the only critical number. Our P' chart is:

x	$(0, 8)$	$(8, \infty)$
$P'(x)$	$-$	$+$
$P(x)$	$\searrow$	$\nearrow$

So by FDTAE, P has an absolute minimum at $x = 8$. That gives $y = 64/8 = 8$.

So yes, the smallest perimeter occurs for an 8×8 square.

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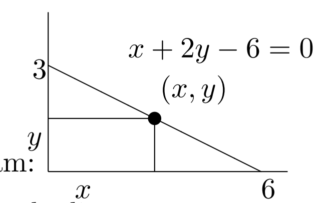
2. A rectangle lies in the first quadrant, with one vertex at the origin, two sides along the coordinate axes, and the fourth vertex on the line $x + 2y - 6 = 0$. Find the maximum area of the rectangle.

Solution. Diagram: 

Since (x, y) lies on the line $x + 2y - 6 = 0$, we have $x = 6 - 2y$.

So the area of the rectangle is $A = xy = (6 - 2y)y = 6y - 2y^2$.

The common sense bounds give $x \geq 0$ and $y \geq 0$, with $x \geq 0$ becoming $6 - 2y \geq 0$, so $2y \leq 6$, i.e., $y \leq 3$. That is, together we have $0 \leq y \leq 3$.

So we must maximize $A(y) = (6 - 2y)y = 6y - 2y^2$ on $[0, 3]$.

We compute $A'(y) = 6 - 4y$, which is always defined. Solving $A' = 0$ gives $4y = 6$, so $y = 3/2$ as the only critical number. Our A' chart is:

y	$[0, 3/2)$	$(3/2, 3]$
$A'(y)$	$+$	$-$
$A(y)$	$\nearrow$	$\searrow$

So by FDTAE, A has an absolute maximum at $y = 3/2$. That gives area $A(3/2) = (6 - 3) \cdot 3/2 = \frac{9}{2}$.

So the maximum area is $\boxed{\frac{9}{2}}$

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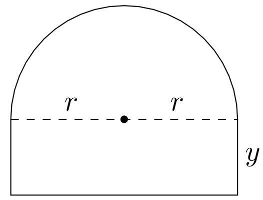
3. A Norman window is a window in the shape of a rectangle with a semicircle on top of it, with diameter the same length as the rectangle. Suppose a Norman window is to have a perimeter of 30 ft. What dimensions will make the area the largest?

Solution. Here's the original picture, with a couple of variable name labels added:



The rectangle part is $2r$ long by y wide, so that part has area $2ry$. The semicircle part is half of a circle of radius r, so its area is $\frac{\pi}{2}r^2$. The total area is therefore $2ry + \frac{\pi}{2}r^2$.

The portion of the perimeter coming from the rectangle is $2r + 2y$, since we have the bottom and two sides of the rectangle, but not the top. The semicircle's portion of the perimeter is $\frac{1}{2} \cdot 2\pi r = \pi r$.

So the total perimeter of the window is $2r + 2y + \pi r = (2 + \pi)r + 2y$, which we set equal to 30. Solving for y, we get

$$(2 + \pi)r + 2y = 30, \quad \text{so} \quad 2y = 30 - (2 + \pi)r, \quad \text{so} \quad y = 15 - \frac{2 + \pi}{2}r.$$

Plugging that into our formula for the area, we want to minimize and maximize

$$A(r) = 2r\left(15 - \frac{2 + \pi}{2}r\right) + \frac{\pi}{2}r^2 = 30r - (2 + \pi)r^2 + \frac{\pi}{2}r^2 = 30r - \left(\frac{4 + \pi}{2}\right)r^2$$

The common sense bounds say that $r \geq 0$ but also that $y \geq 0$. The condition $y \geq 0$ becomes $15 - \frac{2 + \pi}{2}r \geq 0$, so $\frac{2 + \pi}{2}r \leq 15$, so $r \leq \frac{30}{2 + \pi}$.

So we want to minimize and maximize $A(r)$ on the interval $\left[0, \frac{30}{2 + \pi}\right]$.

[Note that since π is a little more than 3, the endpoint $30/(2 + \pi)$ must be a little less than $30/5 = 6$.]

We have $A'(r) = 30 - (4 + \pi)r$, which is **always defined**. Solving $A' = 0$ gives $(4 + \pi)r = 30$, so $r = \frac{30}{4 + \pi}$, which is a little more than 4 and hence is actually in our interval. Our A' chart is

r	$[0, 30/(4 + \pi))$	$(30/(4 + \pi), 30/(2 + \pi)]$
$A'(r)$	$+$	$-$
$A(r)$	$\nearrow$	$\searrow$

So by the First Derivative Test for Absolute Extrema, A has an absolute maximum at $r = \frac{30}{4 + \pi}$ft, which gives

$$y = 15 - \frac{(2 + \pi)}{2} \cdot \frac{30}{(4 + \pi)} = \frac{15}{4 + \pi}\big((4 + \pi) - (2 + \pi)\big) = \frac{30}{4 + \pi}\text{ft}$$

So the biggest window has $\boxed{r = y = \frac{30}{4 + \pi}\text{ft}}$ which is approximately 4.2ft.

[Side note: you can also use CIM to test for max/min, but the computations are pretty nasty at the right endpoint.]

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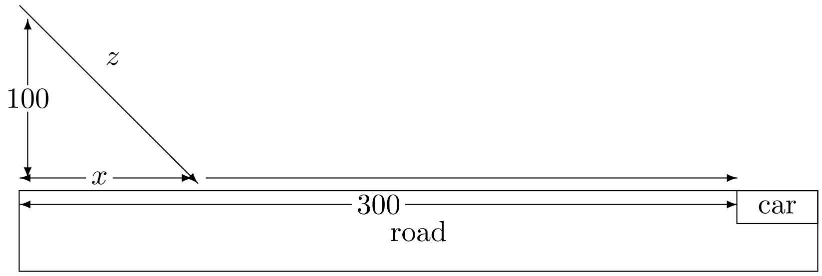
4. A forest ranger can walk through the woods at a rate of 1 m/sec and along a road at 2 m/sec. The ranger is in the woods 100m from the nearest point on a straight road, and he wants to get to a car stopped 300m further down the road. What path should he take to get to the car in the shortest time?

Solution. The ranger's quickest path must consist of a straight line to some point on the road, and then straight down the road:



We have $z = \sqrt{10000 + x^2}$ is the distance the ranger walks through the woods; he then walks $300 - x$ meters down the road. Since time is distance divided by speed, his total time (in seconds) is therefore

$$f(x) = 1 \cdot \sqrt{10000 + x^2} + \frac{1}{2} \cdot (300 - x),$$

which we must minimize for x in the interval $[0, 300]$. We compute

$$f'(x) = \frac{2x}{2\sqrt{10000 + x^2}} - \frac{1}{2} = \frac{2x - \sqrt{10000 + x^2}}{2\sqrt{10000 + x^2}},$$

which is always defined. Solving $f' = 0$ gives $2x = \sqrt{10000 + x^2}$, or (after squaring), $4x^2 = 10000 + x^2$. [Note that x must be positive for equality to have held originally, but we are already forcing $x \in [0, 300]$.]

Thus, $3x^2 = 10000$, so $x = 100/\sqrt{3}$. Applying CIM, testing this and the endpoints, we get

$$f(0) = 100 + 150 = 250; \quad f(300) = 100\sqrt{10} \simeq 316.2, \text{and}$$

$$f(100/\sqrt{3}) = \frac{200}{\sqrt{3}} + 150 - \frac{50}{\sqrt{3}} = 150(1 + \frac{1}{\sqrt{3}}) \simeq 236.6.$$

So the minimum time (of 236.6 seconds) occurs at $x = 100/\sqrt{3}$.

That is, his best path is to walk:

straight to a point $\frac{100}{\sqrt{3}}$ m down the road, and then from there along the road to the car

[Note: we could also use FDTAE to confirm that an absolute minimum occurs at $x = 100/\sqrt{3}$.]

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5. In the forest ranger problem, what is the best path if the car is only 40m down the road instead of 300m?

Solution. The problem is the same as the previous one but with 300 replaced by 40. Thus, we get

$$f(x) = \sqrt{10000 + x^2} + \frac{1}{2} \cdot (40 - x),$$

giving exactly the same

$$f'(x) = \frac{2x - \sqrt{10000 + x^2}}{2\sqrt{10000 + x^2}},$$

which is always defined, and which is zero only at $x = 100/\sqrt{3} \simeq 57.7$. However, this time the interval is $[0, 40]$; so 57.7 is not a critical number.

So applying CIM, we only test the endpoints 0 and 40:

$$f(0) = 120; \quad f(40) = 20\sqrt{29} \simeq 107.7.$$

So the minimum time (of 107.7 seconds) occurs when $x = 40$.

That is, the ranger should walk through the forest directly to the car

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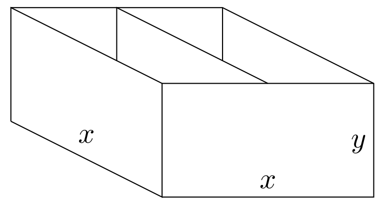
6. We need a cardboard box with a square base, open top, and a vertical partition (wall) inside, parallel to one of the sides. (The partition is also made of cardboard.) If the total volume of the box needs to be $50\,\text{ft}^3$, what dimensions will use the least amount of cardboard?

Solution. Here's the diagram:



The volume of the box is $x^2 y = 50$, so $y = 50x^{-2}$.

The base has area x^2, and the four sides and the partition are each rectangles of area xy. So the total area of cardboard is $x^2 + 5xy$.

The common sense bounds are $x > 0$ and $y > 0$; since $y = 50/x^2$, this is just $x > 0$.

Substituting $y = 50x^{-2}$ in the area formula, we must minimize $A(x) = x^2 + \dfrac{250}{x}$ on $(0, \infty)$.

We compute $A'(x) = 2x - 250x^{-2}$, which is defined everywhere on the domain $(0, \infty)$. Solving $A' = 0$ gives

$$2x = 250x^{-2}, \qquad \text{so} \qquad x^3 = 125, \qquad \text{so} \qquad x = 5.$$

So $x = 5$ feet is the only critical number. Our A' chart is

x	$(0,5)$	$(5,\infty)$
$A'(x)$	$-$	$+$
$A(x)$	$\searrow$	$\nearrow$

So by FDTAE, A has an absolute minimum at $x = 5$. That gives $y = 50/25 = 2$.
So the best box is 5 ft long, 5 ft wide, and 2 ft high.

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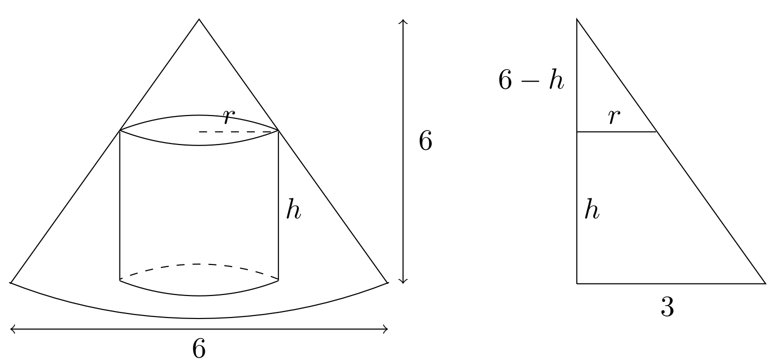
7. Consider a cone such that the height is 6 inches high and its base has diameter 6 in. Inside this cone we inscribe a cylinder whose base lies on the base of the cone and whose top intersects the cone in a circle. What is the maximum volume of the cylinder?

Solution. Diagram:



The second picture shows the cross section of the cone and cylinger, and the similar triangles there show that $\dfrac{r}{3} = \dfrac{6-h}{6}$. Multiplying by 6 gives $2r = 6 - h$, so $h = 6 - 2r$.

The volume of the cylinder is $V = \pi r^2 h = \pi r^2(6 - 2r) = 6\pi r^2 - 2\pi r^3$, and the common sense bounds give $0 \le r \le 3$.

So we must maximize $V(r) = 6\pi r^2 - 2\pi r^3$ on $[0, 3]$.

We have $V'(r) = 12\pi r - 6\pi r^2$, which is always defined. Solving $V' = 0$ gives $6\pi r(2 - r) = 0$, so $r = 0, 2$ are the critical numbers.

Testing the critical numbers and endpoints with CIM:

$$V(0) = 0, \qquad V(3) = 0, \qquad V(2) = \pi(2)^2(6 - 2\cdot 2) = 8\pi$$

So the maximum volume is $\boxed{8\pi \text{ in}^3}$

[**Note**: We could also have confirmed the maximum with FDTAE.]

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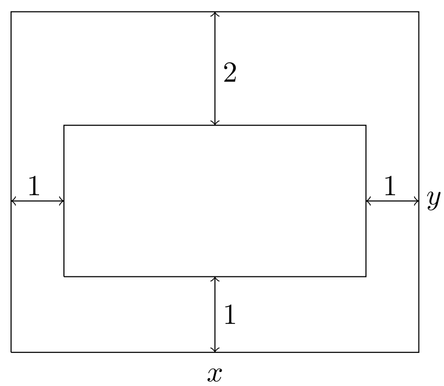
8. A poster is to have an area of 180 square inches, with margins of 1 inch at the bottom and sides, and a margin of 2-inches at the top. What dimensions will give the largest printed area?

Solution. Here's the diagram:



The whole poster is x in long by y in tall, so it has total area xy, which we set equal to 180.

The printed area is $(x - 2)(y - 3)$. Solving $xy = 180$ for y, we obtain that the printed area is

$$A(x) = (x - 2)(180x^{-1} - 3) = 180 - 360x^{-1} - 3x + 6 = 186 - 360x^{-1} - 3x$$

We must have $x \geq 2$ and $y \geq 3$ (to allow for the margins). The second inequality becomes $1800/x \geq 3$, i.e., $x \leq 60$. Thus, we want to minimize $A(x)$ on the interval $[2, 60]$.

We compute $A'(x) = 360x^{-2} - 3$, which is **defined on all of** $[2, 60]$. Solving $A' = 0$ gives $6 = 9600x^{-2}$, so $x^2 = 120$, so $x = \pm\sqrt{120}$. We discard the negative root because it's not in the domain. Our A' chart is:

x	$(2, \sqrt{120})$	$(\sqrt{120}, 60)$
$A'(x)$	$+$	$-$
$A(x)$	$\nearrow$	$\searrow$

So by FDTAE, A has an absolute maximum at $x = \sqrt{120}$ in, or $2\sqrt{30}$ in.

That gives $y = 180/(2\sqrt{30}) = 3\sqrt{30}$ in. So the optimal poster is $\boxed{2\sqrt{30} \text{ in long by } 3\sqrt{30} \text{ in high}}$

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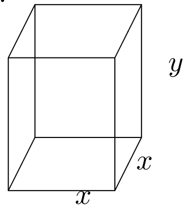
9. A toolshed with a square base and a flat roof is to have volume of 800 cubic feet. If the material for the floor costs \$6 per square foot, the roof \$2 per square foot, and the sides \$5 per square foot, determine the dimensions of the most economical shed.

Solution.

Diagram: 

The volume of the shed is x^2y cubic feet, which we set equal to 800, so $y = 800x^{-2}$.

The floor has area x^2 and hence has cost $6x^2$ dollars.

The roof has area x^2 and hence has cost $2x^2$ dollars.

Each of the four sides has area xy and hence cost $5xy$ dollars.

Summing the floor, roof, and four sides, the total cost is $8x^2 + 20xy$. Substituting $y = 800x^{-2}$, we wish to minimize $C(x) = 8x^2 + 16000x^{-1}$.

The common sense bounds say $x > 0$ and $y > 0$, but $y > 0$ gives nothing new, since $y = 800x^{-2} > 0$ already. So the domain for x is $(0, \infty)$

We have $C'(x) = 16x - 16000x^{-2}$, which is **defined everywhere** on $(0, \infty)$. Solving $C' = 0$ gives $16x = \dfrac{16000}{x^2}$, so $x^3 = 1000$, and hence $x = 10$. Our C' chart is then:

x	$(0, 10)$	$(10, \infty)$
$C'(x)$	$-$	$+$
$C(x)$	$\searrow$	$\nearrow$

So by FDTAE, C has an absolute minimum when $x = 10\,\text{ft}$. This gives $y = 800/10^2 = 8\,\text{ft}$.

So the most economical shed has dimensions **10ft x 10ft x 8ft**

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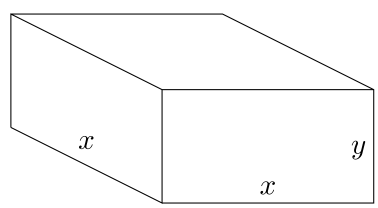
10. A manufacturer wishes to produce rectangular containers with square bottoms and tops, each container having a capacity of 250 cubic inches. The material for the sides costs $2 per square inch for the sides, and the material for the top and bottom costs $4 per square inch. What dimensions of the containers will minimize the cost?

Solution. Here's the diagram:



The volume of the container is x^2y cubic inches, which we set equal to 250, so $y = 250x^{-2}$.

The bottom and top each have area x^2 and hence cost $4x^2$ dollars each, or $8x^2$ dollars together.

Each of the four sides has area xy, so each costs $2xy$, or $8xy$ dollars together.

Summing the six faces, the total cost of each container is $8x^2 + 8xy = 8x^2 + 8x(250x^{-2}) = 8x^2 + 2000x^{-1}$.

The common sense bounds say $x > 0$ and $y > 0$, but $y > 0$ gives nothing new. So we must minimize $C(x) = 8x^2 + 2000x^{-1}$ on the domain $(0, \infty)$.

We have $C'(x) = 16x - 2000x^{-2}$, which is **defined everywhere** on $(0, \infty)$. Solving $C' = 0$ gives $16x = \dfrac{2000}{x^2}$, so $x^3 = 125$, and hence $x = 5$. Our C' chart is then:

x	$(0, 5)$	$(5, \infty)$
$C'(x)$	$-$	$+$
$C(x)$	$\searrow$	$\nearrow$

So by FDTAE, C has an absolute minimum when $x = 5\,\text{in}$. This gives $y = 250/5^2 = 10\,\text{in}$.

So the most economical containers have dimensions **5in x 5in x 10in**

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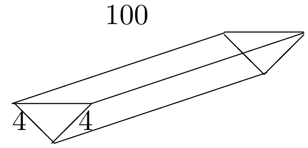
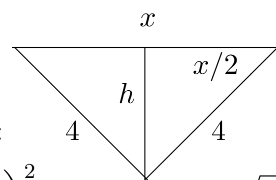
11. A rectangular sheet of metal 8 inches wide and 100 inches long is folded along the center to form a triangular trough. Two triangular pieces of metal are attached to the ends of the trough, and the trough is to be filled with water:



(a). How deep should the trough be to maximize the capacity of the trough?

(b). What is this maximum possible capacity?

Solution. Here's a diagram of the cross section:



By the Pythagorean Theorem, we have $h^2 + \left(\frac{x}{2}\right)^2 = 4^2$, i.e., $h = \sqrt{16 - \frac{x^2}{4}}$.

So the area of the triangle above is $A = \frac{1}{2} h \cdot x = \frac{x}{2}\sqrt{16 - \frac{x^2}{4}}$, so the volume of the trough is

$V(x) = 100A = 50x\sqrt{16 - \frac{x^2}{4}}$.

The common sense bounds are $0 \le x \le 8$, so we must maximize V on the domain $[0, 8]$.

We compute

$$V'(x) = 50\left(16 - \frac{x^2}{4}\right)^{1/2} + 50x \cdot \frac{1}{2}\left(16 - \frac{x^2}{4}\right)^{-1/2} \cdot \left(-\frac{x}{2}\right) = \frac{25}{2}\left(16 - \frac{x^2}{4}\right)^{-1/2}\left[4\left(16 - \frac{x^2}{4}\right) - x^2\right]$$

$$= \frac{25}{2}\left(16 - \frac{x^2}{4}\right)^{-1/2}[64 - 2x^2] = -25\left(16 - \frac{x^2}{4}\right)^{-1/2}(x^2 - 32)$$

which is defined on $[0, 8)$ but undefined at $x = 8$ (because there the expression under the square root is zero). Solving $V' = 0$ gives $x = \pm\sqrt{32}$. Discarding the negative root, which is not in the interval, the only critical points in our interval are at $x = \sqrt{32}, 8$. Our V' chart is:

x	$(0, \sqrt{32})$	$(\sqrt{32}, 8)$
$V'(x)$	$+$	$-$
$V(x)$	$\nearrow$	$\searrow$

So by FDTAE, the absolute max occurs at $x = \sqrt{32} = 4\sqrt{2}\,\text{in}$, and this maximum is

$$V(\sqrt{32}) = 50\sqrt{32}\sqrt{16-8} = 50 \cdot 4\sqrt{2} \cdot \sqrt{8} = 200 \cdot 4 = 800\,\text{in}^3$$

(Alternatively, using CIM, testing the endpoints gives $V(0) = V(8) = 0$ and so $V(\sqrt{32}) = 800$ is the max.)

Also, $x = \sqrt{32}$ gives $h = \sqrt{16-8} = \sqrt{8} = 2\sqrt{2}\,\text{in}$.

So the optimal trough has $\boxed{\text{depth } 2\sqrt{2}\,\text{in and volume } 800\,\text{in}^3}$

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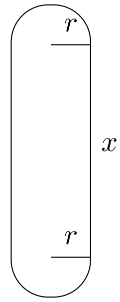
12. An outdoor track is to be created in the shape shown and is to have perimeter of 440 yards. Find the dimensions for the track that maximize the area of the rectangular portion of the field enclosed by the track.

Solution. Here's the diagram:



The perimeter is $2x + 2\pi r$, which we set equal to 440, so that $x = \dfrac{440 - 2\pi r}{2} = 220 - \pi r$.

The area of the rectangular portion is $(2r) \cdot x = 440r - 2\pi r^2$, and the common sense bounds say $x, r \geq 0$. Note that $x \geq 0$ gives $r \leq 200/\pi$, so

$$\text{We must maximize } A(r) = 440r - 2\pi r^2 \text{ on } [0, \tfrac{220}{\pi}].$$

We compute $A'(r) = 440 - 4\pi r$, which is always defined. Solving $A' = 0$ gives $r = \dfrac{110}{\pi}$ as our only critical number. Our A' chart is:

r	$(0, 110/\pi)$	$(110/\pi, 220/\pi)$
$A'(r)$	$+$	$-$
$A(r)$	$\nearrow$	$\searrow$

So by FDTAE, the absolute max occurs at $r = \dfrac{110}{\pi}$, which gives $x = 220 - \pi\left(\dfrac{110}{\pi}\right) = 110$.

So the optimal dimensions are $\boxed{x = 110 \text{ yards}}$ and $\boxed{r = \dfrac{110}{\pi} \text{ yards}}$

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13. Show that the entire region enclosed by the outdoor track in the previous problem has maximum area if the track is circular.

Solution. With the same diagram and variables as in the previous problem, we still have $x = 220 - \pi r$.

This time, however, the area we want is the area enclosed by the entire track, which is
$pir^2 + 2rx = \pi r^2 + 2r(220 - \pi r) = \pi r^2 + 440r - 2\pi r^2 = 440r - \pi r^2$

The common sense bounds still say $x, r \geq 0$ and hence that $0 \leq r \leq \dfrac{200}{\pi}$. Note that $x \geq 0$ gives $r \leq 200/\pi$, so

$$\text{We must maximize } A(r) = 440r - \pi r^2 \text{ on } \left[0, \frac{220}{\pi}\right].$$

We compute $A'(r) = 440 - 2\pi r$, which is always defined. Solving $A' = 0$ gives $r = \dfrac{220}{\pi}$ as our only critical number.

[Since that's not in the middle of our interval, we use CIM.] Testing, while writing $A(r) = r\Big(440 - \pi r\Big)$, gives:

$$A(0) = 0, \qquad A\left(\frac{220}{\pi}\right) = \frac{220}{\pi}\Big(440 - 220\Big) = \frac{220^2}{\pi}$$

So the absolute maximum occurs at $r = \dfrac{220}{\pi}$, which gives $x = 220 - \pi r = 0$. That is, the optimal track has $x = 0$, which is a $\boxed{\text{circular}}$ track.

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14. Find a function $f(x)$ that satisfies $f''(x) = 12x^2 + 5$ and $f'(1) = 5$, and which passes through the point $(1, 3)$.

Solution. We have $f'(x) = 4x^3 + 5x + C$, so $5 = f'(1) = 4 + 5 + C$, which gives $C = -4$. So $f'(x) = 4x^3 + 5x - 4$.

We then have $f(x) = x^4 + \frac{5}{2}x^2 - 4x + D$. Since the graph passes through $(1, 3)$, we have $f(1) = 3$, so $3 = f(1) = 1 + \frac{5}{2} - 4 + D$, so $D = \frac{7}{2}$. Thus, $\boxed{f(x) = x^4 + \frac{5}{2}x^2 - 4x + \frac{7}{2}}$

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15. Find a function $f(x)$ that satisfies $f''(x) = x + \sin x$, $f'(0) = 6$, and $f(0) = 4$.

Solution. We have $f'(x) = \frac{1}{2}x^2 - \cos x + C$, so $6 = f'(0) = 0 - 1 + C$, and hence $C = 7$. That is, $f'(x) = \frac{1}{2}x^2 - \cos x + 7$.

We then have $f(x) = \frac{1}{6}x^3 - \sin x + 7x + D$, so $4 = f(0) = D$. Thus, $\boxed{f(x) = \frac{1}{6}x^3 - \sin x + 7x + 4}$

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16. Find a function $f(x)$ that satisfies $f''(x) = 2 - 12x$, $f(0) = 9$, and $f(2) = 15$.

Solution. We have $f'(x) = -6x^2 + 2x + C$, so $f(x) = -2x^3 + x^2 + Cx + D$, for some constants C and D.

So $9 = f(0) = D$, and then $15 = f(2) = -2(8) + 4 + 2C + 9$, so that $15 = 2C - 3$, i.e., $2C = 18$, so $C = 9$.

That is, $\boxed{f(x) = -2x^3 + x^2 + 9x + 9}$

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17. Find a function $f(x)$ that satisfies $f''(x) = 20x^3 + 12x^2 + 4$, $f(0) = 8$ and $f(1) = 5$.

Solution. We have $f'(x) = 5x^4 + 4x^3 + 4x + C$, so $f(x) = x^5 + x^4 + 2x^2 + Cx + D$.

So $8 = f(0) = D$, and then $5 = f(1) = 1 + 1 + 2 + C + 8$, so $C = -7$.

That is, $\boxed{f(x) = x^5 + x^4 + 2x^2 - 7x + 5}$

Curve Sketching For each of the following functions, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

18. $f(x) = x^3 - 3x^2 + 3x + 10$

Solution.
Domain: $f(x)$ has domain $(-\infty, \infty)$

VA: f is a polynomial, so no vertical asymptotes

HA: There are no horizontal asymptotes for this f since it's a nonconstant polynomial.
[In fact, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.]

First Derivative Information:
$f'(x) = 3x^2 - 6x + 3$, which is always defined. Setting $f' = 0$ gives

$$3x^2 - 6x + 3 = 3(x - 1)(x - 1) = 0 \Longrightarrow x = 1$$

So $x = 1$ is the only critical number. Our f' chart is

x	$(-\infty,1)$	$(1,\infty)$
$f'(x)$	$+$	$+$
$f(x)$	$\nearrow$	$\nearrow$

So f is increasing on $(-\infty,1)$ and $(1,\infty)$, with no extreme values.

Second Derivative Information:
$f''(x) = 6x - 6$, which is always defined. Setting $f'' = 0$ gives $x = 1$. Our f'' chart is

x	$(-\infty,1)$	$(1,\infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\bigcap$	$\bigcup$

So f is concave down on $(-\infty,1)$ and concave up on $(1,\infty)$, with an inflection point at $x = 1$.

See separate PDF for a sketch.

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19. $f(x) = \dfrac{3x^2}{1-x^2}$

Solution.
Domain: The denominator is 0 for $x = \pm 1$, but otherwise f is defined. So the domain is $\{x \,|\, x \neq \pm 1\}$.

VA: Vertical asymptotes at $x = \pm 1$ (where denom= 0)

HA: We compute $\lim\limits_{x\to\pm\infty} \dfrac{3x^2}{1-x^2} \cdot \dfrac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim\limits_{x\to\pm\infty} \dfrac{3}{\frac{1}{x^2} - 1} = -3$

So f has a horizontal asymptote of $y = -3$ on both sides.

First Derivative Information:
We compute $f'(x) = \dfrac{6x(1-x^2) - 2x(3x^2)}{(1-x^2)^2} = \dfrac{6x}{(1-x^2)^2}$, which is defined on the whole domain of f.

Solving $f' = 0$ gives $x = 0$ as the only critical number. Our f' chart, taking into account critical numbers and vertical asymptotes, is:

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$f'(x)$	$-$	$-$	$+$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

So f is decreasing on $(-\infty,-1)$ and $(-1,0)$ and increasing on $(0,1)$ and $(1,\infty)$. Moreover, f has a local minimum at $x = 0$.

Second Derivative Information:
We compute $f''(x) = \dfrac{6(1-x^2)^2 - 6x\cdot 2(1-x^2)\cdot(-2x)}{(1-x^2)^4} = \dfrac{6(1-x^2)+24x^2}{(1-x^2)^3} = \dfrac{6(1+3x^2)}{(1-x^2)^3}$, which is defined on the whole domain of f and is never zero. Our f'' chart, taking into account vertical asymptotes, is:

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\bigcap$	$\bigcup$	$\bigcap$

So f is concave down on $(-\infty,-1)$ and$(1,\infty)$ and concave up on $(-1,1)$, with no inflection points.

See separate PDF for a sketch.

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20. $f(x) = \dfrac{x}{x-2}$

Solution.
Domain: All $x \neq 2$.

VA: Vertical asymptote at $x = 2$, because of the denominator.

HA: We compute $\lim_{x\to\pm\infty} \frac{x}{x-2} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\pm\infty} \frac{1}{1-\frac{2}{x}} = 1.$
So f has a horizontal asymptote at $y = 1$ on both sides.

First Derivative Information:
We compute $f'(x) = \frac{1(x-2) - 1(x)}{(x-2)^2} = \frac{-2}{(x-2)^2}$, which is defined on the whole domain of f.

Solving $f' = 0$ gives no critical numbers. Our f' chart, taking into account the asymptote, is

x	$(-\infty, 2)$	$(2, \infty)$
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$

So f is decreasing on both $(-\infty, 2)$ and $(2, \infty)$, with no extreme values.

Second Derivative Information:

Meanwhile, $f''(x) = \frac{0(x-2)^2 - (-2) \cdot 2(x-2) \cdot 1}{(x-2)^4} = \frac{4}{(x-2)^3}$, which is defined on the whole domain of f and is never zero. Our f'' chart is

x	$(-\infty, 2)$	$(2, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\bigcap$	$\bigcup$

So f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$, with no inflection points.

See separate PDF for a sketch.

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21. $f(x) = 2x^3 + 5x^2 - 4x$

Solution.
Domain: $f(x)$ has domain $(-\infty, \infty)$

VA: f is a polynomial, so no vertical asymptotes

HA: There are no horizontal asymptotes for this f since it's a nonconstant polynomial.
[In fact, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.]

First Derivative Information:
We have $f'(x) = 6x^2 + 10x - 4$, which is always defined. Setting $f' = 0$ gives

$$2(3x^2 + 5x - 2) = 0, \qquad \text{, i.e.,} \qquad 2(3x-1)(x+2) = 0$$

so $x = 1/3$ and $x = -2$ are the only critical numbers. Our f' chart is

x	$(-\infty, -2)$	$(-2, 1/3)$	$(1/3, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\infty, -2)$ and on $(1/3, \infty)$; and f is decreasing on $(-2, 1/3)$. Moreover, f has a local max at $x = -2$ and a local min at $x = 1/3$

• Second Derivative Information:
We have $f''(x) = 12x + 10$, which is always defined. Setting $f'' = 0$ gives $12x = -10$, so $x = -5/6$. Our f'' chart is

x	$(-\infty, -5/6)$	$(-5/6, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\bigcap$	$\bigcup$

So f is concave down on $(-\infty, -5/6)$ and concave up on $(-5/6, \infty)$, with an inflection point at $x = -5/6$.

See separate PDF for a sketch.

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22. $f(x) = 3x^4 + 4x^3$

Solution.
Domain: $f(x)$ has domain $(-\infty, \infty)$

VA: f is a polynomial, so no vertical asymptotes

HA: There are no horizontal asymptotes for this f since it's a nonconstant polynomial.
[In fact, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.]

First Derivative Information:

We have $f'(x) = 12x^3 + 12x^2$, which is always defined. Setting $f' = 0$ gives $12x^2(x+1) = 0$, and hence $x = 0, -1$. Our f' chart is:

x	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$

So f is increasing on $(-1, \infty)$; and f is decreasing on $(-\infty, -1)$, with a local min at $x = -1$.

Second Derivative Information:

We have $f''(x) = 36x^2 + 24x$, which is always defined. Setting $f'' = 0$ gives $12x(3x + 2) = 0$, so $x = 0, -2/3$. Our f'' chart is

x	$(-\infty, -2/3)$	$(-2/3, 0)$	$(0, \infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-2/3, 0)$ and concave up on $(-\infty, -2/3)$ and $(0, \infty)$, with inflection points at $x = 0$ and $x = -2/3$.

See separate PDF for a sketch.

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23. $f(x) = x^4 - 6x^2$

Solution.
Domain: $f(x)$ has domain $(-\infty, \infty)$

VA: f is a polynomial, so no vertical asymptotes

HA: There are no horizontal asymptotes for this f since it's a nonconstant polynomial.
[In fact, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.]

First Derivative Information:

We have $f'(x) = 4x^3 - 12x$, which is always defined. Setting $f' = 0$ gives $4x(x^2 - 3) = 0$, so $x = 0, \pm\sqrt{3}$. Our f' chart is

x	$(-\infty, -\sqrt{3})$	$(\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\sqrt{3}, 0)$and $(\sqrt{3}, \infty)$; and f is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Moreover, f has a local max at $x = 0$ and local mins at $x = \pm\sqrt{3}$.

Second Derivative Information:

We have $f''(x) = 12x^2 - 12$, which is always defined. Setting $f'' = 0$ gives $12(x-1)(x+1) = 0$, so $x = \pm 1$. Our f'' chart is

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$f''(x)$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\bigcup$	$\bigcap$	$\bigcup$

So f is concave down on $(-1,1)$ and concave up on $(-\infty,-1)$ and $(1,\infty)$, with inflection points at $x=\pm 1$.

See separate PDF for a sketch.

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24. $f(x)=\dfrac{3x^5-20x^3}{32}$

Solution.
Domain: $f(x)$ has domain $(-\infty,\infty)$

VA: f is a polynomial, so no vertical asymptotes

HA: There are no horizontal asymptotes for this f since it's a nonconstant polynomial.
[In fact, $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$.]

First Derivative Information:
We have $f'(x)=\frac{1}{32}(15x^4-60x^2)$, which is always defined. Setting $f'=0$ gives $15x^2(x^2-4)=0$, so $x=0,\pm 2$. Our f' chart is

x	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(0,\infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\infty,-2)$ and on $(2,\infty)$; and f is decreasing on $(-2,2)$. Moreover, f has a local max at $x=-2$ and a local min at $x=2$.

Second Derivative Information:
We have $f''(x)=\frac{1}{32}(60x^3-120x)$, which is always defined. Setting $f''=0$ gives $60x(x^2-2)=0$, so $x=0,\pm\sqrt{2}$. Our f'' chart is

x	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},\infty)$
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\bigcap$	$\bigcup$	$\bigcap$	$\bigcup$

So f is concave down on $(-\infty,-\sqrt{2})$ and $(0,\sqrt{2})$, and concave up on $(-\sqrt{2},0)$ and$(\sqrt{2},\infty)$. Moreover, f has inflection points at $x=0$ and $x=\pm\sqrt{2}$.

See separate PDF for a sketch.

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25. $f(x)=\dfrac{1}{x^2-9}$

Solution.
Domain: The denominator is 0 for $x=\pm 3$, but otherwise f is defined. So the domain is $\{x \mid x\neq \pm 3\}$

VA: Vertical asymptotes at $x=\pm 3$ (where denom$=0$)

HA: We compute $\lim_{x\to\pm\infty}\dfrac{1}{x^2-9}\cdot\dfrac{x^{-2}}{x^{-2}}=\lim_{x\to\pm\infty}\dfrac{x^{-2}}{1-9x^{-2}}=\dfrac{0}{1-0}=0.$
So f has a horizontal asymptote of $y=0$ on both sides.

First Derivative Information:
We have $f'(x)=\dfrac{0(x^2-9)-1(2x)}{(x^2-9)^2}=\dfrac{-2x}{(x^2-9)^2}$, which is defined on the whole domain of f.

Solving $f' = 0$ gives $x = 0$ as the only critical number. Our f' chart is:

x	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	$+$	$+$	$-$	$-$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$

So f is increasing on $(-\infty, -3)$ and $(-3, 0)$; and f is decreasing on $(0, 3)$ and $(3, \infty)$. Moreover, f has a local max at $x = 0$.

Second Derivative Information:
We have $f''(x) = \dfrac{-2(x^2-9)^2 - (-2x)\cdot 2(x^2-9)(2x)}{(x^2-9)^4} = \dfrac{-2(x^2-9)+8x^2}{(x^2-9)^3} = \dfrac{6x^2+18}{(x^2-9)^3}$, which is defined on the whole domain of f and is never zero. Our f'' chart is

x	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-3, 3)$ and concave up on $(-\infty, -3)$ and $(3, \infty)$, with no inflection points.

See separate PDF for a sketch.

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26. $f(x) = \dfrac{2x^3 + 45x^2 + 315x + 600}{x^3}$. And **before you start**:
take my word for it $f'(x) = \dfrac{-45(x+4)(x+10)}{x^4}$ and $f''(x) = \dfrac{90(x+5)(x+16)}{x^5}$.

Solution.
Domain: the denominator is 0 for $x = 0$, but otherwise f is defined. So the domain is $\{x \mid x \neq 0\}$.

VA: vertical asymptote at $x = 0$ (where denom= 0)

HA: We compute $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} 2 + \dfrac{45}{x} + \dfrac{315}{x^2} + \dfrac{600}{x^3} = 2$.
So f has a horizontal asymptote of $y = 2$ on both sides.

First Derivative Information:

Meanwhile, f' is defined everywhere on the domain of f. Setting $f' = 0$ gives $x = -4, -10$. Our f' chart is:

x	$(-\infty, -10)$	$(-10, -4)$	$(-4, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

So f is increasing on $(-10, -4)$, and decreasing on $(-\infty, -10)$, on $(-4, 0)$, and on $(0, \infty)$. Moreover, there is a local minimum at $x = -10$ and a local maximum at $x = -4$.

Second Derivative Information:

f'' is also defined everywhere on the domain of f. Setting $f'' = 0$ gives $x = -16$ and $x = -5$. The f'' chart is:

x	$(-\infty, -16)$	$(-16, -5)$	$(-5, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cup$

So f is concave up on $(-16, -5)$ and $(0, \infty)$, and it is concave down on $(-\infty, -16)$ and $(-5, 0)$. There are inflection points at $x = -16$ and $x = -5$.
See separate PDF for a sketch.

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27. Compute $\lim_{x\to\infty} \dfrac{x^3+1}{x^7+2x^{7/2}}$

Solution. $\lim_{x\to\infty} \dfrac{x^3+1}{x^7+2x^{7/2}} = \lim_{x\to\infty} \dfrac{x^3+1}{x^7+2x^{7/2}} \cdot \dfrac{(\frac{1}{x^7})}{(\frac{1}{x^7})} = \lim_{x\to\infty} \dfrac{\frac{1}{x^4}+\frac{1}{x^7}}{1+\frac{2}{x^{7/2}}} = \dfrac{0}{1} = \boxed{0}$

---

28. Compute $\lim_{x\to\infty} \dfrac{x^6+1}{x^3+9x^2+7}$

Solution. $\lim_{x\to\infty} \dfrac{x^6+1}{x^3+9x^2+7} = \lim_{x\to\infty} \dfrac{x^6+1}{x^3+9x^2+7} \cdot \dfrac{(\frac{1}{x^3})}{(\frac{1}{x^3})} = \lim_{x\to\infty} \dfrac{x^3+\frac{1}{x^3}}{1+\frac{9}{x}+\frac{7}{x^3}} = \dfrac{\infty}{1} = \boxed{\infty}$

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29. Compute $\lim_{x\to\infty} \dfrac{2x+1}{\sqrt{9x^2+5}}$

Solution. $\lim_{x\to\infty} \dfrac{2x+1}{\sqrt{9x^2+5}} = \lim_{x\to\infty} \dfrac{2x+1}{\sqrt{9x^2+5}} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{2+x^{-1}}{\sqrt{9x^2+5}\cdot\sqrt{x^{-2}}} = \lim_{x\to\infty} \dfrac{2+x^{-1}}{\sqrt{9+5x^{-2}}}$

$= \dfrac{2+0}{\sqrt{9+0}} = \dfrac{2}{\sqrt{9}} = \boxed{\dfrac{2}{3}}$

---

30. Compute $\lim_{x\to-\infty} \dfrac{2x+1}{\sqrt{9x^2+5}}$

Solution. $\lim_{x\to-\infty} \dfrac{2x+1}{\sqrt{9x^2+5}} = \lim_{x\to-\infty} \dfrac{2x+1}{\sqrt{9x^2+5}} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \dfrac{2+x^{-1}}{\sqrt{9x^2+5}\cdot(-\sqrt{x^{-2}})} = \lim_{x\to-\infty} -\dfrac{2+x^{-1}}{\sqrt{9+5x^{-2}}}$

$= -\dfrac{2+0}{\sqrt{9+0}} = -\dfrac{2}{\sqrt{9}} = \boxed{-\dfrac{2}{3}}$

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31. Use Riemann Sums to estimate $\int_0^1 x^2+1\,dx$ using 4 equal-length subintervals and right endpoints.

Solution. We have $\Delta x = (1-0)/4 = 1/4$, and $x_i = i/4$. So with $f(x) = x^2+1$, the Riemann sum is:

$R_4 = f\left(\frac{1}{4}\right)\cdot\frac{1}{4} + f\left(\frac{2}{4}\right)\cdot\frac{1}{4} + f\left(\frac{3}{4}\right)\cdot\frac{1}{4} + f\left(\frac{4}{4}\right)\cdot\frac{1}{4} = \frac{1}{4}\left[\left(\frac{1}{16}+1\right)+\left(\frac{4}{16}+1\right)+\left(\frac{9}{16}+1\right)+\left(\frac{16}{16}+1\right)\right]$

$= \frac{1}{4}\left[4+\frac{30}{16}\right] = 1+\frac{15}{32} = \boxed{\frac{47}{32}}$

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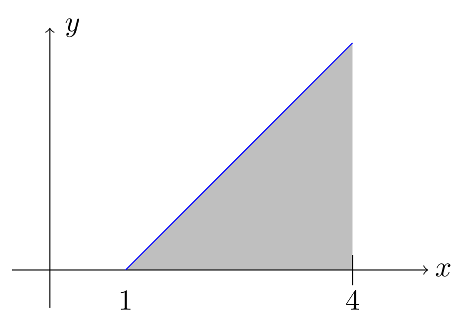
32. Compute $\int_1^4 x-1\,dx$ using three different methods:

(a) using the Area interpretation of the definite integral.
(b) using the Fundamental Theorem of Calculus.
(c) from the limit definition, i.e., using Riemann Sums.

Solution. (a) Here's a sketch:



The region is completely above the x-axis, so the integral is the area, which is

$\frac{1}{2}\text{base}\cdot\text{height} = \frac{1}{2}(3)(3) = \boxed{\frac{9}{2}}$

(b) We have $\int_1^4 x - 1\ dx = \frac{x^2}{2} - x\Big|_1^4 = \left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right) = 8 - 4 - \frac{1}{2} + 1 = 5 - \frac{1}{2} = \boxed{\frac{9}{2}}$

(c) Use $f(x) = x - 1$, with $a = 1$ and $b = 4$. Then $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, and $x_i = a + i\Delta x = 1 + \frac{3i}{n}$.

So the n-th Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right) \cdot \left(\frac{3}{n}\right) = \sum_{i=1}^{n} \left(\frac{3i}{n}\right) \cdot \frac{3}{n} = \sum_{i=1}^{n} \frac{9i}{n^2}$$

$$= \frac{9}{n^2}\sum_{i=1}^{n} i = \frac{9}{n^2} \cdot \frac{n(n+1)}{2} = \frac{9}{2}\left(1 + \frac{1}{n}\right)$$

Thus, $\int_1^4 x - 1\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{9}{2}\left(1 + \frac{1}{n}\right) = \frac{9}{2} \cdot 1 = \boxed{\frac{9}{2}}$

33. Evaluate $\int_{-1}^{1} x\ dx$ using Riemann Sums.

Solution. Use $f(x) = x$, with $a = -1$ and $b = 1$. Then $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$, and $x_i = a + i\Delta x = -1 + \frac{2i}{n}$.

So the n-th Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right) \cdot \left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[-1 + \frac{2i}{n}\right] \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{-2}{n} + \sum_{i=1}^{n} \frac{4i}{n^2}$$

$$= -\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i = -\frac{2}{n}(n) + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = -2 + 2\left(1 + \frac{1}{n}\right) = \frac{2}{n}.$$

Thus, $\int_{-1}^{1} x\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n} = \boxed{0}$

34. Evaluate $\int_0^2 x^2 - 5x\ dx$ using Riemann Sums.

Solution. Use $f(x) = x^2 - 5x$, with $a = 0$ and $b = 2$. Then $\Delta x = \frac{2-(0)}{n} = \frac{2}{n}$, and $x_i = a + i\Delta x = \frac{2i}{n}$.

So the n-th Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 - 5\left(\frac{2i}{n}\right)\right] \cdot \frac{2}{n} = \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} - \frac{10i}{n}\right] \cdot \frac{2}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{8i^2}{n^3} - \frac{20i}{n^2}\right) = \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{20}{n^2}\sum_{i=1}^{n} i = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{20}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 10\left(1 + \frac{1}{n}\right) = \frac{4}{3}(2) - 10 = \frac{8}{3} - \frac{30}{3} = \boxed{-\frac{22}{3}}$$

Compute the following definite and indefinite integrals.

35. $\int_{\pi/6}^{\pi/2} \csc^2 t\, dt$

Solution. $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 t\,dt = -\cot t\Big|_{\pi/6}^{\pi/2} = -\frac{\cos(\pi/6)}{\sin(\pi/6)} + \frac{\cos(\pi/2)}{\sin(\pi/2)} = -\frac{\sqrt{3}/2}{1/2} + \frac{0}{1} = \boxed{-\sqrt{3}}$

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36. $\displaystyle\int \frac{(2w^2+1)(w-3)}{\sqrt{w}}\,dw$

Solution. $\displaystyle\int \frac{(2w^2+1)(w-3)}{\sqrt{w}}\,dw = \int w^{-1/2}(2w^3 - 6w^2 + w - 3)\,dw$

$\displaystyle = \int 2w^{5/2} - 6w^{3/2} + w^{1/2} - 3w^{-1/2}\,dw = \boxed{\frac{4}{7}w^{7/2} - \frac{12}{5}w^{5/2} + \frac{2}{3}w^{3/2} - 6w^{1/2} + C}$

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37. $\displaystyle\int x^5 + \frac{1}{x^5} + \sqrt[5]{x} + \frac{x}{5}\,dx$

Solution. $\displaystyle\int x^5 + \frac{1}{x^5} + \sqrt[5]{x} + \frac{x}{5}\,dx = \int x^5 + x^{-5} + x^{1/5} + \frac{1}{5}x\,dx = \boxed{\frac{1}{6}x^6 - \frac{1}{4}x^{-4} + \frac{5}{6}x^{6/5} + \frac{1}{10}x^2 + C}$

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38. $\displaystyle\int_1^4 \frac{4}{x^3} - \frac{3}{\sqrt{x}}\,dx$

Solution.

NOTE: This problem originally appeared in the printout as $\displaystyle\int_{-4}^{-1} \frac{4}{x^3} - \frac{3}{\sqrt{x}}\,dx$, which is undefined because the integrand doesn't make sense for x in the interval $[-4,-1]$ of integration.

Here is a solution to the correct problem:

$$\int_1^4 \frac{4}{x^3} - \frac{3}{\sqrt{x}}\,dx = \int_1^4 4x^{-3} - 3x^{-1/2}\,dx = -2x^{-2} - 6x^{1/2}\Big|_1^4 = \left(-\frac{2}{16} - 6\sqrt{4}\right) - \left(-\frac{2}{1} - 6\sqrt{1}\right)$$

$$= -\frac{1}{8} - 6\cdot 2 + 2 + 6 = -\frac{1}{8} - 4 = \boxed{-\frac{33}{8}}$$

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39. $\displaystyle\int (\sec\theta + \tan\theta)\sec\theta\,d\theta$

Solution. $\displaystyle\int (\sec\theta + \tan\theta)\sec\theta\,d\theta = \int \sec^2\theta + \sec\theta\tan\theta\,d\theta = \boxed{\tan\theta + \sec\theta + C}$