Math 111-01, Fall 2023 Professor Rob Benedetto

Solutions Midterm Exam 2

1. **(16 points; 2 parts)** Compute the following derivatives by any legal method. Simplify your answers.

(a) $f'(x)$, where $f(x) = x^3 \sec(4-5x)$

(b) The second derivative $R''(x)$, where $R(x) = \dfrac{x^3 - 6x + 8}{\sqrt{x}}$

Solutions. (a): $f'(x) = 3x^2 \sec(4-5x) + x^3 \sec(4-5x)\tan(4-5x) \cdot (-5) =$
$\boxed{3x^2 \sec(4-5x) - 5x^3 \sec(4-5x)\tan(4-5x)}$

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(b): $R(x) = x^{5/2} - 6x^{1/2} + 8x^{-1/2}$, so $R'(x) = \frac{5}{2}x^{3/2} - 3x^{-1/2} - 4x^{-3/2}$, and hence
$\boxed{R''(x) = \frac{15}{4}x^{1/2} + \frac{3}{2}x^{-3/2} + 6x^{-5/2}}$

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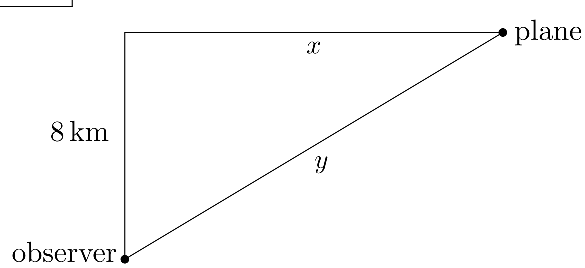
2. **(20 points)** A jet plane, flying in a straight line at a constant altitude of 8 km, passes directly over an observer on the ground. When the jet has flown a further 6 km (i.e., 6 km past the point on its path that was directly above the observer), its speed is $\frac{2}{3}$ km/sec. At that moment, how fast is the (diagonal) distance from the observer to the jet increasing?

Solution. Here's the **Picture**:



Variables:
x =horizontal distance plane is from the spot directly above observer, in km
y =diagonal distance, in km

(And t =time, in sec)

Main **Equation**: $y^2 = 8^2 + x^2$

Differentiate (implicitly, w.r.t. time): $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$

Use key moment info:

At the key moment, we have $x = 6$ km and we are told that $\dfrac{dx}{dt} = \dfrac{2}{3}$km/sec

Also, plugging $x = 6$ into the original equation gives $y^2 = 64 + 36 = 100$, so $y = \pm 10$. Given the picture, we have $y > 0$, so $y = 10$ km.
Plugging $x = 6$, $y = 10$ and $\dfrac{dx}{dt} = \dfrac{2}{3}$ into the derivative equation above,

we have $2(6)\frac{2}{3} = 2(10)\frac{dy}{dt}$ i.e., $\frac{dy}{dt} = \boxed{\frac{2}{5}\text{ km/sec}}$

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3. **(20 points)** Let $F(x) = \dfrac{x+4}{x^2+9}$.
Find the absolute minimum and absolute maximum values of F on the interval $[-4, 4]$.

Solution. (By Closed Interval Method):
$F'(x) = \dfrac{1(x^2+9)-(x+4)(2x)}{(x^2+9)^2} = \dfrac{x^2+9-2x^2-8x}{(x^2+9)^2} = \dfrac{-x^2-8x+9}{(x^2+9)^2} = \dfrac{-(x+9)(x-1)}{(x^2+9)^2}$,
which is **always defined**.

Solving $F' = 0$ gives $x = -9, 1$, but -9 is not in the interval.

So the only critical number is $x = 1$. Testing it and endpoints:

$$F(-4) = \frac{0}{16+9} = 0, \quad F(1) = \frac{5}{1+9} = \frac{1}{2}, \quad F(4) = \frac{8}{16+9} = \frac{8}{25}$$

So the $\boxed{\text{absolute maximum is } \frac{1}{2}}$ and the $\boxed{\text{absolute minimum is } 0}$

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4. **(15 points)** Consider the equation: $x^4 + 2y^4 = x^3y + xy^3 + 5$
Find the equation of the tangent line to this curve at the point $(1, -1)$.

Solution. By Implicit Diff: $4x^3 + 8y^3y' = 3x^2y + x^3y' + y^3 + 3xy^2y'$,

So at $(x, y) = (1, -1)$, we have $4 - 8y' = -3 + y' - 1 + 3y'$.

That is, $4 - 8y' = -4 + 4y'$, which becomes $12y' = 8$, so $y' = \frac{2}{3}$.

Thus, by point-slope, the tangent line is $y + 1 = \frac{2}{3}(x-1)$, i.e., $y + 1 = \frac{2}{3}x - \frac{5}{3}$, so $\boxed{y = \frac{2}{3}x - \frac{5}{3}}$

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5. **(15 points)** Suppose $f(x)$ is a function with the property that

$$f(1) = -1, \qquad f'(1) = 2, \qquad f(5) = -6, \qquad \text{and} \qquad f'(5) = 7.$$

Let $G(x) = f(x^3 + 4x^2)$, and let $H(x) = [f(x)]^3 + 4[f(x)]^2$. Compute $G'(1)$ and $H'(1)$.

Solution. We have $G'(x) = f'(x^3 + 4x^2) \cdot (3x^2 + 8x)$, so $G'(1) = f'(5) \cdot (3 + 8) = 7 \cdot 11 = \boxed{77}$

We have $H'(x) = 3f(x)^2 f'(x) + 8f(x)f'(x)$,
so $H'(1) = 3f(1)^2 f'(1) + 8f(1)f'(1) = 3(-1)^2 \cdot 2 + 8(-1) \cdot 2 = 6 - 16 = \boxed{-10}$

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6. **(14 points)** Let $H(x) = \sin^2(4x) + \tan(3x) - 5\cos(2x)$. Compute $H'\left(\frac{\pi}{12}\right)$. Simplify your answer.

Solution. We have $H'(x) = 2\sin(4x)\cos(4x) \cdot 4 + \sec^2(3x) \cdot 3 - 5(-\sin(2x)) \cdot 2$,

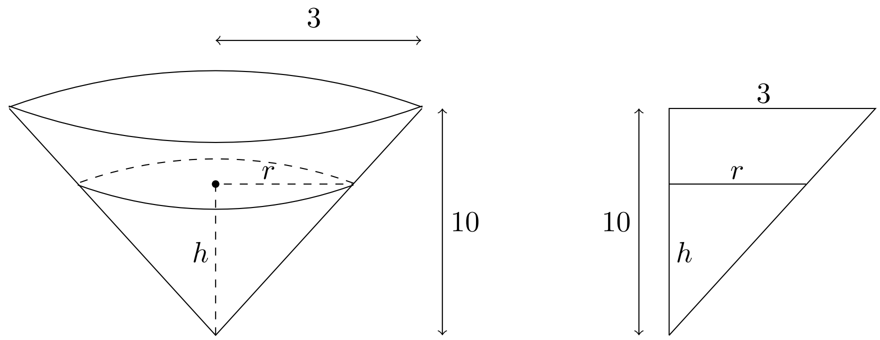
so $H'\left(\frac{\pi}{12}\right) = 8\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) + 3\sec^2\left(\frac{\pi}{4}\right) + 10\sin\left(\frac{\pi}{6}\right)$

$= 8 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + 3\left(\frac{1}{\sqrt{2}/2}\right)^2 + 10 \cdot \frac{1}{2}$

$= 2\sqrt{3} + 3 \cdot 2 + 5 = \boxed{2\sqrt{3} + 11}$

OPTIONAL BONUS A. (2 points.) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). The cup is being filled with water, and at the moment when the depth of the water is 5 cm, the water level is rising at 1/2 cm/sec. How fast is water being poured into the cup at that moment?

Solution. **Picture**:



Variables:
r = radius of **water** in tank (cm)
h = depth of **water** in tank (cm)
V = volume of **water** in tank (cm^3)
[Key moment: $h = 5\text{cm}$ and $\frac{dh}{dt} = 1/2\text{cm/sec}$]

Equations: $V = \frac{\pi}{3}r^2h$ and (by similar triangles) $\frac{h}{10} = \frac{r}{3}$, so $r = \frac{3h}{10}$, so $V = (\pi/3)r^2h = (3\pi/100)h^3$.

Deriv: $\frac{dV}{dt} = \frac{3\pi}{100} \cdot 3h^2\frac{dh}{dt}$.

Solve: At key moment, with $h = 5$ and $dh/dt = 1/2$: $\frac{dV}{dt} = \frac{9\pi \cdot 25}{200} = \boxed{\frac{9\pi}{8}\text{cm}^3/\text{sec}}$

OPTIONAL BONUS B. (1 point.) This past weekend, there was a presidential election in a certain country. Since no candidate received a majority of the votes, by that country's laws, there will be a runoff in November between the top two candidates: one with the last name Massa, and the other with the last name Milei. In what country did this election occur?

Answer. Argentina